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# A Theoretical Treatment of the Sliding of Glaciers in the Absence of Cavitation

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# A THEORETICAL TREATMENT OF THE SLIDING OF GLACIERS IN THE ABSENCE OF CAVITATION

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A model is proposed for the description of glacier sliding which includes the nonlinearity of the flow law for ice. The model describes coupled flow problems in the basal ice and a thin water film, together with a temperature problem in the underlying bedrock. To determine the sliding law relating basal velocity to basal stress, the sliding theory should be formulated as a boundary layer to the larger-scale bulk ice flow.

Dimensional analysis indicates that the regelative component of ice velocity may be neglected, provided roughness is absent at the smallest wavelengths, and then the ice flow effectively uncouples from the other problems. In this case, with the crucial (but unrealistic) assumption that the flow law for temperate ice is independent of the moisture content, there exist complementary variational principles that describe the functional form of the sliding law and give bounds on the magnitude of the 'roughness' coefficient. These principles are valid for nonlinear stress-strain rate relations and for

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non-vanishing bedrock corrugation, and indicate how the basal velocity is determined by *two* parameters that together describe the degree of roughness of the bed. Specific estimates are then given.

Finally, the main weakness in the model as a predictor of quantitatively accurate results is pointed out: that is, that the variation of moisture within the basal layer, and the resultant effect on the flow law, are neglected. A valid description of this phenomenon does not yet appear to be available.

## 1. INTRODUCTION

It is a well known fact that the basal ice of a temperate glacier can slide over the underlying bedrock. This is achieved by means of a lubricating water film at the ice–rock interface, which is maintained there by pressure melting on the upstream faces of protruding obstacles. In this case an appropriate boundary condition at the bedrock for the ice flow is that the tangential stress in the ice is zero (since the viscosity of water is, by comparison, negligible) but, from the point of view of the motion of the bulk of the ice mass, a more relevant ‘boundary condition’ is to prescribe the ‘basal velocity’ as a function of the effective drag due to the resistance offered to the motion by corrugations in the bedrock. Such a ‘boundary condition’ is usually called the ‘sliding law’, and much effort has gone into determining its form since the pioneering work of Weertman (1957).

Many of the important physical processes have been identified, in particular the (possibly crucial) phenomenon of cavitation (Lliboutry 1968), but many of the theoretical models presented have not been properly formulated, and in many cases mathematical procedures have been abandoned in favour of apparently arbitrary assumptions; in view of this, the validity of the results should be treated with some caution. The only process that can be said to be properly understood is the sliding of a Newtonian fluid over a wavy bedrock, incorporating regelative effects (Nye 1969, Morland 1976*a*). Non-Newtonian effects and cavitation have only been considered previously in an empirical manner.

In this paper the sliding theory is considered from the point of view of determining an effective boundary condition for the bulk ice flow. This can be done formally by using the ideas of matched asymptotic expansions (Cole 1968); the problem then becomes one of determining the flow of ice in a basal ‘boundary layer’ adjacent to the bedrock, and there are associated problems for the flow in the water film, and for the temperature in the bedrock.

In §3 (nomenclature is included as §2) the detailed physics in these regions is discussed and a brief review of the literature is given; in §4 it is specified how the ice flow problem is to be formulated in terms of the large-scale glacial flow. This makes precise what we mean by such ill-defined terms as ‘basal velocity’ and ‘basal shear stress’. Furthermore, the sliding law we seek to establish will be in terms of the appropriate dimensionless units for the ‘outer’ flow, and thus we shall be able to see at a glance how the magnitude of the basal velocity is determined by the various dimensionless parameters that occur.

In §5 the complete set of equations and boundary equations to be solved is set out; these are then scaled, and it is shown how the problem in the water film is uncoupled from those in the ice and the rock, which are coupled by the regelation process. Much of this scaling procedure is not new, but it is retained in full here, since (i) it gives the appropriate scalings when the nonlinear flow law due to Glen (1955) is considered, and (ii) it provides the basis for an explicit description of the water film.

For convenience, the coupled set of scaled and non-dimensionalized equations describing the flow and temperature problems in the basal ice and bedrock, respectively, is repeated at the end of § 5. With the stated physical assumptions, this set is a valid model in the absence of cavitation (when the ice separates from the bedrock).

An exact solution for this problem is beyond our means, even in the (possibly dubious) asymptotic limit of small bedrock roughness (which has been considered for a Newtonian flow by others, e.g. Nye (1969)). However, all that we require is a relation between the basal velocity and the resistance to the motion offered by the bedrock, and hence it is not necessary to obtain explicitly the complete solution. We can obtain estimates for this relation by examining a variational principle, and, accordingly, an appropriate principle for the model equations given in § 5 is stated in § 6. Since the principle is actually an equality when the trial functions are the solution, we immediately obtain a dimensionless estimate for the magnitude of the velocity. One then sees that this magnitude is crucially dependent on the mean slope of the (rough) bedrock; furthermore (in contrast to the results of other authors), steady-state velocities of any magnitude can be predicted merely by varying this slope within realistic limits.

The specific estimates we obtain for the sliding law are based on the assumption that the regelative component of the bedrock resistance is negligible. This assumption is motivated by the scalings in § 5, which suggest that the normal velocity at the ice–water interface, due to regelation, is very small, except past obstacles of dimension less than about 1 mm. (This compares with a value for the ‘controlling wavelength’ in Newtonian flow of 7.7 cm (Morland 1976*a*)). The analysis in § 6 is therefore based on the supposition that roughness is absent at wavelengths of less than about 1 mm. This seems a reasonable hypothesis.

In § 7 it is shown how to construct trial functions for the variational principle, and explicit bounds are obtained in a very simple example.

In § 8 we reconsider our previous neglect of regelation; this is only justified if bedrock roughness is absent on a sufficiently small scale. Since the regelative length-scale (controlling wavelength) is much smaller than the dimensions of the overall bedrock corrugation, we could model the regelative effect as a tangential drag imposed at the bedrock on a non-regelative flow of ice. In this model, we can still use the bounds obtained in § 6 for the drag, and we find that inclusion of such a regelative traction combines (to lowest order) *additively* with the basal stress produced by the ice flow over the bedrock. No obvious method of determining the magnitude of such a regelative drag suggests itself.

In § 9, we compare the results of the model with certain experimentally observed features. It becomes evident that the proposed model requires a certain amount of modification. In particular, the effect of moisture contained in the ice on its viscosity may be of crucial importance, but it is unclear at present how such effects should be satisfactorily modelled.

The conclusions of the investigation are presented in § 10.

## 2. NOMENCLATURE

*Suffixes*

- I properties of ice  
o 'outer' flow variables

*Symbols*

- $A$  proportionality factor in temperate ice flow law, (5.4*b*)  
 $a$  constant defined by (7.18)  
 $C$  constant in sliding law, (3.1) (also (6.74))  
 $d$  typical glacier depth  
 $e_{ij}$  strain rate tensor, (5.4*b*)  
 $e$  second strain rate invariant, (5.4*b*)  
 $\tilde{e}$  scaled form of  $e$ , (5.87)  
 $\tilde{e}_{ij}$  scaled form of  $e_{ij}$   
 $e^*$  scaled form of  $e$ , (6.61)  
 $f_i$  component of gravitational force, (6.7)  
 $f(x)$  defined by (7.7)  
 $F(Y)$  defined by (7.13)  
 $F(\tau_b, T)$  temperature-dependent sliding law, (9.2)  
 $g$  acceleration due to gravity  
 $g'$  component of acceleration due to gravity normal to mean bedrock slope  
 $G$  geothermal heat flux  
 $g(x)$  defined by (7.9)  
 $h_S$  smooth component of bedrock profile  
 $h_R$  rough component of bedrock profile  
 $h_D$  actual dimensional bedrock profile  
 $\mathcal{H}, \mathcal{H}_r$  functionals for stress variational principle, (6.22)  
 $\mathcal{H}_2$  second variation of  $\mathcal{H}_r$ , (6.44)  
 $H$  dimensionless glacier depth  
 $h$  equals  $h_R$   
 $h_*$  scaled mean quadratic bedrock slope  
 $h_1$  dimensional depth measured perpendicularly to the line of mean bedrock slope (9.1)  
 $J$  functional defined by (6.12)  
 $J_V$  functional for stress principle, (6.40)  
 $J_0$  value of  $J$  at solution to ice flow problem, (6.26)  
 $J_2$  second variation of  $J_V$ , (6.41)  
 $K$  curvature, (5.70)  
 $k$  thermal conductivity, (5.12); constant in trial function, (7.15)  
 $k_R$  thermal conductivity of rock  
 $k_1, k_2$  constants in trial function, (7.1)  
 $k(x)$  defined in (7.11)  
 $l$  glacier length scale  
 $L$  latent heat of melting of ice

$M$	period of the rough bedrock $h$
$m, \hat{m}$	constants in flow law, (6.28)
$n$	exponent in Glen's flow law for temperate ice, (5.4 <i>b</i> )
$\mathbf{n}$	unit normal vector at the ice–bedrock interface
$p$	pressure
$p_w$	pressure within the water film
$p_A$	atmospheric pressure
$\tilde{p}$	scaled pressure within basal ice flow, (5.20)
$P$	scaled pressure within water film, (5.47)
$\mathbf{q} = (q_1, q_2)$	ice flow velocity
$R$	roughness parameter, (6.86)
$r$	ratio of ice and rock thermal conductivities, (5.80)
$S$	Nye's water film space variable, (5.70)
$S_b$	bedrock boundary
$S_\infty$	boundary of basal ice flow, located in the matching region between inner and outer flows
$S(x)$	dimensionless ice–water interface
$T$	temperature; dimensionless water film thickness, (5.70)
$T^*$	dimensionless, scaled ice temperature, (5.48)
$[T]$	temperature scale in ice and bedrock
$\mathbf{t}$	unit tangential vector at ice–rock interface
$T_M$	melting temperature of ice at atmospheric pressure
$u_b$	basal sliding velocity
$u$	$x$ -component of ice velocity
$U_i$	components of prescribed velocity on $S_\infty$ , (6.13)
$U_o$	velocity scale for outer flow, (5.16)
$u_o$	dimensionless outer $x$ -velocity component
$\tilde{u}$	dimensionless scaled inner $\tilde{x}$ -velocity, (5.20)
$[u]$	water film velocity scale, (5.38)
$U$	dimensionless water film $X$ -velocity
$u_b^*$	dimensionless, scaled, $O(1)$ sliding velocity, (6.53)
$U_{\text{shear}}$	scale of velocity change in the outer flow due to shearing
$v$	$y$ -component of ice velocity
$V$	bounding volume for variational integral; dimensionless, scaled water film velocity, (5.38)
$v_i$	components of ice velocity
$\tilde{v}$	dimensionless, scaled inner $\tilde{y}$ -velocity, (5.20)
$V_M$	dimensionless, scaled melting velocity, (5.35)
$V_b$	bedrock volume beneath $S_b$
$x$	coordinate along line of mean bedrock slope
$x_o$	dimensionless outer $x$ coordinate
$[x]$	$x$ -scale of rough bedrock boundary
$X_{\text{av}}$	averaging length for constructing $h_S$ , (4.4)
$x_Z$	point where basal ice reaches the pressure melting point
$x_M$	point where melting surface 'breaks away' from basal flow layer

$\tilde{x}$	dimensionless inner flow $x$ -coordinate
$X$	dimensionless, scaled $x$ -coordinate in water film; Nye's lateral space variable, (5.70)
$y$	coordinate perpendicular to $x$ coordinate
$[y]$	scale of undulations in $h_R$
$y_M$	melting surface
$y^*$	ordinate of matching region, (6.3)
$y_o$	dimensionless outer $y$ -coordinate
$\tilde{y}$	dimensionless inner $y$ -coordinate
$Y$	dimensionless, scaled water film $y$ -coordinate, (5.29); equals $y - \nu h$ , (7.2)
$\alpha$	dimensionless measure of the regelative component of ice velocity, (5.35); Nye's tangential angle (5.70)
$\alpha(x)$	defined in (6.69)
$\Gamma$	Lliboutry's (1976) regelation parameter
$\Gamma(\ell_{rs}), \hat{\Gamma}(\tau_{rs})$	flow law functions defined by (6.9) and (6.21)
$\delta_{ij}$	Kronecker delta (equals unity if $i = j$ and zero if $i \neq j$ )
$\delta v_i, \delta \tau_{ij}$	variations of the stated variables from the solutions for the basal ice flow
$\delta J$	first variation of $J$
$\delta$	parameter measuring the shallowness of the glacier
$\delta^*$	parameter measuring the thickness of the water film
$\epsilon$	mean bedrock slope
$\theta$	Clausius–Clapeyron constant, (5.13); stress trial function, (7.2)
$\kappa_w$	thermal diffusivity of water, (5.7)
$\lambda_*$	Morland's transition wavelength, (5.59)
$\Lambda_*$	dimensionless geothermal heat flux, (5.84)
$\mu$	dimensionless parameter measuring the deviation of the surface slope from the mean bedrock slope, (5.24a)
$\mu_w$	viscosity of water
$\nu$	bedrock corrugation, (5.17)
$\rho$	density of ice
$\rho_w$	density of water
$\sigma_{ij}$	stress tensor
$\sigma$	bedrock asperity, (5.18)
$\Sigma$	dimensionless, scaled water film thickness, (5.31)
$\tau_b$	basal stress
$\tau_{ij}$	stress deviator tensor
$\tau_1, \tau_2$	longitudinal and tangential components of stress deviator tensor, (5.4a)
$\tau$	second invariant of stress tensor, (5.4b)
$\tilde{\tau}$	scaled form of $\tau$ , (5.87)
$\tilde{\tau}_{ij}$	scaled form of $\tau_{ij}$
$\tau_s$	tangential traction at bedrock, (6.11)
$[\tau]_o$	stress scale for outer flow
$\tilde{\tau}_1, \tilde{\tau}_2$	scaled forms of $\tau_1$ and $\tau_2$ for inner flow
$\tau^*$	scaled form of $\tau$ , (6.62)
$\bar{\tau}_s$	average traction, (6.76)



$\tau_s^*$	scaled form of $\tau_s$ , (6.80)
$\bar{\tau}_s^*$	scaled form of $\bar{\tau}_s$ , (6.81)
$\phi$	Airy stress function defined by (6.56)
$\phi^*$	scaled form of $\phi$ , (6.59)
$\chi$	angle of inclination of glacier surface to horizontal
$\psi$	stream function defined by (6.57); Nye's dimensionless constant (5.70)
$\psi^*$	scaled form of $\psi$ , (6.58)
$\psi_1$	defined by (6.67)

### 3. DESCRIPTION AND SPECIFICATION OF THE PROBLEM

We shall consider the flow of temperate ice (that is, ice at the pressure melting temperature) over a rough bedrock. As the ice flows round a typical protuberance, its pressure melting temperature changes according to the Clausius–Clapeyron law (we neglect any effects of shear stress on the melting point) and, since the pressure is greater on the upstream side of the obstacle, the ice temperature is correspondingly lower there. This temperature difference induces a heat flux in the bedrock that is sufficient to melt a thin film of water adjacent to the bedrock. The heat flux in the bedrock towards (away from) the upstream (downstream) side of the obstacle is reinforced by a heat flux in the ice, due to the variation of melting temperature with pressure, and the fact that the pressure in the ice increases (decreases) as the bedrock is approached upstream (downstream) of the obstacle.

The reason for assuming the ice is everywhere temperate is discussed below. In this case, the temperature is described fully by the Clausius–Clapeyron equation, and the role of the energy equation is to describe the amount of moisture present in the ice (Lliboutry 1976). If the viscosity of ice is considered to be a function of its moisture content (Lliboutry 1976), then a description of the moisture content is a necessary constituent of the solution. The equations to be solved are then much more difficult, and will not be considered in this paper.

Now although the bedrock heat flux must cause a thin lubrication film to form, owing to regelation, there is of course no guarantee that such a film will cover the entire bed; in fact it would be rather surprising if it did. We may therefore expect there to be, in general, patches of basal ice where there is no lubrication film; it is not clear in this case what the appropriate kinematic boundary conditions should be. One is, of course, that there be no normal velocity, but the other is not necessarily that there be no tangential stress on the ice. We might suppose that the ice at such patches would require a small but finite traction to be applied in order that a small temporary film appear so that the ice there could slide briefly before equilibrium was restored. Such patches, though not ‘cold’, would correspond to the cold patches of Robin (1976), would contribute to stick–slip (‘stictional’) motion, and as explained by Robin, would appear to offer one possible mechanism for the formation of roches moutonnées. Note also that such patches *would* be cold in a zone of ‘sub-temperate’ sliding (explained below; see also Fowler (1979)). Stick–slip motion due to such frictional patches could be satisfactorily modelled over the time scales of interest by the application of some mean traction applied at the bedrock. (This may also be a valid method of incorporating additional drag due to basal debris (Morland 1976*b*) and also of modelling regelative effects, see §8.)

The problem of description of the water film is known to cause inconsistencies in the theory of regelation (Nye 1973), and a similar difficulty besets the sliding of glaciers. A more complete



discussion of this inconsistency (which has not yet been resolved) is included below (p. 655). In this paper, we will *choose* to ignore this aspect of the problem, thus assuming that the bedrocks considered are permissible in Morris's (1976) sense. In essence, this means that the (periodic) bedrock is 'nearly' sinusoidal, in the sense that its Fourier components decrease rapidly in amplitude. A more realistic treatment would include a solution of this aspect of the problem, but our point of view is that it is more profitable (and possible) to analyse first the different aspects of sliding separately, before attempting to conjoin different results. As we show, we can to some extent interpret the lubrication film inconsistency as representing a form of cavitation.

Weertman (1957) was the first to give a quantitative theory of glacier sliding. He considered the flow of ice over an idealized bed consisting of a regular array of cubic obstacles on a flat plane. For a given shear stress  $\tau_b$  applied to an ice flow over such a bedrock, he estimated the velocities due to pure regelation and pure viscous flow (we avoid the use of the phrase 'enhanced plastic flow' since the flow is not plastic), and found them to be proportional to  $\tau_b$  and  $\tau_b^n$  respectively, where  $n$  is the exponent in the well known flow law for ice (Glen 1955). He was led to the concept of a controlling obstacle size, and thence to an approximate intermediate law for the basal velocity  $u_b$ :

$$u_b = C\tau_b^{\frac{1}{2}(n+1)}. \quad (3.1)$$

Weertman later refined his ideas (for example, Weertman 1964) by considering a more realistic bedrock with obstacles of varying size, and introduced the idea of cavitation behind obstacles. However, his basic approach remains non-mathematical, and numerical values of  $C$  in (3.1) should be treated with some caution; nevertheless, Weertman has since defended his ideas (1971).

Lliboutry is the other major exponent of sliding theory. In a long paper (Lliboutry 1968) he reviews previous work and proposes his own theory. In this he envisages ice sliding over a two-dimensional bedrock (of small slope), and introduces the effect of cavitation. His method is, like Weertman's, semi-theoretical (there is much use of physically motivated approximation), but nevertheless it represents a useful first attempt. In particular, he found that inclusion of cavitation led to a multivalued function for the velocity  $u_b$  in terms of the stress  $\tau_b$ . Such a result, if justified, would have a profound influence on the large-scale dynamics of glaciers; indeed, it is the feeling of the author that such multivaluedness may be an essential constituent of the mechanism of surges (Meier & Post 1969).

Nye (1969, 1970) and Kamb (1970) independently took a more mathematical viewpoint. They considered the slow flow of a Newtonian fluid over a slowly varying bedrock, with a suction velocity at the bed due to melting and refreezing that may be found from a calculation of the temperature fields in ice and bedrock.

(Actually, if the ice is considered to be fully temperate, only the bedrock temperature problem need be solved, since then the ice temperature is already prescribed by means of the Clausius–Clapeyron relation. As already explained, the temperate ice energy equation is then used to determine the moisture content. However, both Nye and Kamb, and later Morland (1976*a*), solve an energy equation for the ice of the form

$$\nabla^2 T = 0, \quad (3.2)$$

justifying neglect of convective terms by the statement that they are generally small, although this is not usually true, except for relatively small sliding velocities (less than about  $5 \text{ m a}^{-1}$ ). In fact, (3.2) is correct for temperate ice in the Newtonian case (only), since then the pressure  $p$  also satisfies Laplace's equation; however, this is not so when the fluid is non-Newtonian.)

Nye's and Kamb's theories are subject to the criticism of not being properly mathematically formulated, as the induced resistance on the bedrock is not balanced by an imposed stress at infinity (upwards into the ice). In fact, the results are valid by virtue of the fact that the bedrock roughness slope (here called  $\nu$ ) is taken to be small, so that the velocity perturbation about the basal velocity (a first-order effect) induces a stress at infinity as a second-order effect. If  $\nu$  were taken to be  $O(1)$ , no solution would exist of the problem as formulated by Nye. Morland (1976*a*) recasts Nye's theory using the methods of complex variables, and includes the glacier depth explicitly. This eliminates previous errors, but it is felt that his approach, which essentially involves solving the equations of motion in the entire ice mass, is not appropriate for the formulation of the sliding law as a 'boundary condition' for the main glacial flow. He develops his ideas in a further paper (Morland 1976*b*) in which he includes a tangential traction at the bedrock as a model for the additional resistance due to basal debris transported with the ice.

In his paper, Kamb (1970) developed an approximate solution for a non-Newtonian flow on the basis that the ice viscosity was a function of the vertical coordinate only. With this apparently arbitrary assumption, he obtains Weertman's intermediate law (3.1) for a white bedrock, that is, one which has the same aspect at all wavelengths less than the roughness scale. A similar law has been obtained by Lliboutry (1975, 1976), who also claims that the numerical value of  $C$  in (3.1) is too small to account for sliding velocities larger than about  $10 \text{ m a}^{-1}$ . Kamb obtains

$$u_b \propto \tau_b^n \quad (3.3)$$

for a 'truncated' white bedrock (one with roughness absent at the smaller wavelengths).

Lliboutry's (1976) argument is worth commenting on. He states that the only dimensional parameters occurring in a bedrock ice flow are  $u_b$ ,  $\tau_b$ ,  $A$  (constant in Glen's flow law) and a regelation parameter  $\Gamma$ . From this he deduces by dimensional similarity that the flow law must be of the Weertman form

$$u_b = C(A/\Gamma) \tau_b^{\frac{1}{2}(n+1)}. \quad (3.4)$$

This simple and attractive argument would be correct if the properly formulated problem indeed depended only on the four stated constants. Unfortunately, the notion of a 'basal' ice flow requires some specification of 'where' the base is, and we shall see that this involves the glacier depth  $d$  in the problem formulation. If we consider  $d$ , there is then no such thing as a completely 'dimensionless' bedrock, since there must be a maximum amplitude  $[y]$  in its variation about the 'mean' bedrock (and we require  $[y] \ll d$ ). Thus a proper formulation involves more parameters, and Lliboutry's conclusion is untenable.

The aim of this paper is to formulate the sliding law as a dimensionless boundary condition for the equations of motion of a large-scale flow of ice over some 'mean' bedrock: a model for such a flow is presented elsewhere (Fowler & Larson 1978, hereafter referred to as I). Let us therefore turn to a consideration of how best to formulate the problem from this point of view.

We shall suppose that the bedrock profile is composed of a smooth component  $h_s$  and a rough component  $h_r$  (cf. Nye 1970). It is intuitively obvious how such a decomposition can be made in reality, though the mathematical process is slightly more subtle. We then expect that the bulk flow of the glacier follows the mean profile  $h_s$  (changes in  $h_s$  affect the whole depth of the glacier), whereas the rough component  $h_r$  only affects the flow in a thin basal layer close to the bedrock. With these ideas, it is clear that the total flow may be represented by two components, in the manner of boundary-layer theory (Batchelor 1967). The 'outer' flow satisfies dimensionless equations scaled with large scale parameters; in particular the natural height and length scales

for this problem are  $d$  and  $l$ , where  $d$  is a typical depth and  $l$  may be taken as the length of the glacier (I). On the other hand, the ‘inner flow’ near the bedrock follows the contours of the local bedrock roughness  $h_R$ , and has a natural length scale of the dimension of the roughness. The formal procedure of relating these two solutions is obtained by requiring that they ‘match’ into each other in some intermediate region (which is ‘far’ from  $h_R$  on the inner scale, but ‘near’  $h_S$  on the outer scale). This matching procedure (described by Cole (1968)) suffices to determine an effective boundary condition on  $h_S$  for the outer flow, which is precisely the sliding law. ‘The’ basal velocity  $u_b$  is then defined as the (apparent) limiting value of the tangential velocity in the outer flow at  $h_S$  and the basal stress is similarly defined. The details of this process are described by Fowler (1979).

#### 4. FORMULATION OF THE PROBLEM

Following I, we consider a two-dimensional glacial ice flow down an inclined bedrock surface of mean slope  $\epsilon$ ; and we take axes  $(x, y)$  along and perpendicular to the line of mean slope. Now let us suppose that the dimensional bedrock surface  $h_D$  may be written as the sum of two components,

$$h_D = dh_S \left( \frac{x}{l} \right) + [y] h_R \left( \frac{x}{[x]} \right). \quad (4.1)$$

Here,  $d$  and  $l$  (as already mentioned) are the height and length scales of the outer flow problem, and similarly  $[x]$  and  $[y]$  are appropriate length scales for the inner flow problem. We assume these scales are such that the dimensionless bedrock profiles  $h_S$  and  $h_R$  satisfy

$$h_S, h'_S, h_R, h'_R \lesssim O(1), \quad (4.2)$$

where a prime denotes differentiation with respect to the argument of the function. The constraints on  $h_R$  motivate the choice of  $[x]$  and  $[y]$ , and the first constraint on  $h_S$  follows from the second provided the origin is chosen accordingly. Choosing  $h'_S \lesssim O(1)$  is the criterion under which the scaled model presented in I remains valid, and this assumption will be adopted for convenience in this paper, although in fact we only require the scales  $d$  and  $l$  in (4.1) to be such that

$$[x], [y] \ll d, \quad [x], [y] \ll l, \quad (4.3)$$

and thus the present theory would be formally valid for sliding in ice-falls, or over relatively rough patches of  $h_S$ .

We may formally define  $h_S$  by constructing a running average of  $h_D$  (actually  $h_D/d$ ) over a distance  $X_{av}$  such that

$$[x] \ll X_{av} \ll l. \quad (4.4)$$

For example, with  $l = 10$  km,  $[x] = 5$  m, a suitable averaging distance would be  $X_{av} \approx 200$  m. This procedure is described by Nye (1970). There is no unique choice for  $h_S$ , but it is clear that  $h_S$  is (formally) uniquely defined as  $X_{av}/l \rightarrow 0$  and  $[x]/X_{av} \rightarrow 0$ . In practice,  $h_S$  is only affected by  $O(X_{av}/l)$  and  $h_R$  by  $O([x]/X_{av})$  if  $X_{av}$  is allowed to vary within the prescribed limit (4.4). It should be clear that such variation will have a negligible effect on the sliding law.

It was shown by Fowler (1977, 1979) that there is a basal region (there denoted by  $(x_Z, x_M)$ ) in which the ice is at the pressure melting point (and the full temperate sliding law is supposed to hold) but the ice above (in the outer flow region) is cold. At  $x_M$ , the melting surface  $y_M$  dividing cold and temperate ice ‘breaks away’ from the bedrock  $h_S$  into the outer flow. From the point of view of our boundary-layer ideas, this means that the melting surface within the basal layer tends

to infinity (i.e. leaves the basal layer) at  $x_M$ . Since the bedrock is temperate in  $(x_Z, x_M)$ , the melting surface leaves the bedrock at  $x_Z$ , and thus, as  $x$  increases from  $x_Z$  to  $x_M$ , it gradually rises through the basal layer, as shown in figure 1. In  $x > x_M$ ,  $y_M$  'emerges' into the outer flow, and then, formally, the basal layer is entirely temperate, and the Clausius–Clapeyron relation holds everywhere.

With the foregoing physics in mind, let us now consider the specific model equations for the sliding of ice within the basal layer.

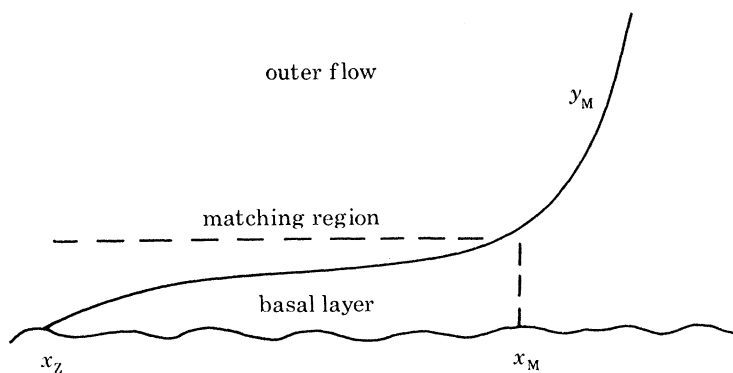


FIGURE 1. Matching surface within the basal layer.

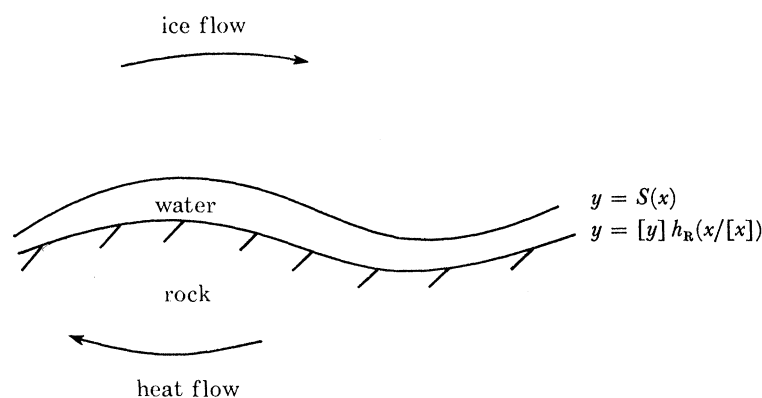


FIGURE 2. Basal flow geometry.

## 5. MODEL EQUATIONS AND BOUNDARY CONDITIONS

We consider the geometry shown in figure 2. Since we are interested in flow on the  $[x]$ -scale,  $h_S$  is effectively constant, and without loss of generality we take the local coordinate system such that  $h_S = 0$ . We also suppose that the roughness  $h_R$  is a periodic function. This is an artificial device introduced so that the mean stress induced at  $y = \infty$  by the flow does not vary with  $x$ , and is a legitimate construction since the basal stress  $\tau_b$  and velocity  $u_b$  (being outer flow variables) are considered to change negligibly over distances of  $o(l)$ .

The equations of motion for the ice are

$$\nabla \cdot \mathbf{q} = 0, \quad (5.1)$$

$$p_x = \rho g' \epsilon + \tau_{1x} + \tau_{2y}, \quad (5.2)$$

$$p_y = -\rho g' + \tau_{2x} - \tau_{1y}, \quad (5.3)$$

where subscripts  $x$  and  $y$  indicate partial derivatives,  $\epsilon$  is the mean bedrock slope ( $\arctan \epsilon$  is the angle of inclination of the  $x$ -axis to the horizontal),  $\rho$  is the density of the ice,  $g' = g(1 + \epsilon^2)^{-\frac{1}{2}}$ ,  $g$  is the acceleration due to gravity,  $\mathbf{q} \equiv (q_1, q_2) \equiv (u, v)$  is the velocity,  $p$  is the pressure, and  $\tau_1$  and  $\tau_2$  are the longitudinal and tangential stress deviators, defined for an incompressible fluid by

$$\left. \begin{aligned} \sigma_{ij} &= -p\delta_{ij} + \tau_{ij}, \\ \tau_1 &= \tau_{11} = -\tau_{22}, \\ \tau_2 &= \tau_{12} = \tau_{21}, \end{aligned} \right\} \quad (5.4a)$$

where  $\sigma_{ij}$  is the stress tensor,  $\delta_{ij}$  is the Kronecker delta, and subscripts 1 and 2 refer to  $x$  and  $y$  components respectively. We further suppose that the ice behaves according to Glen's flow law, that is,

$$\left. \begin{aligned} e_{ij} &= A\tau^{n-1}\tau_{ij}, \\ e &= A\tau^n, \quad n \approx 3, \\ 2e^2 &= e_{ij}e_{ij}, \quad 2\tau^2 = \tau_{ij}\tau_{ij}, \\ e_{ij} &= \frac{1}{2} \left( \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} \right), \end{aligned} \right\} \quad (5.4b)$$

where, for temperate ice,  $A$  may be expected to be a function of the moisture content (Lliboutry 1976), but will here be considered to be a constant. The usual summation convention is employed in (5.4b) and hereafter.

The equations of motion and energy in the water film are

$$\nabla \cdot \mathbf{q} = 0, \quad (5.5)$$

$$\rho(\mathbf{q} \cdot \nabla) \mathbf{q} + \nabla[p + \rho g' y - \epsilon \rho g' x] = \mu_w \nabla^2 \mathbf{q}, \quad (5.6)$$

$$\mathbf{q} \cdot \nabla T = \kappa_w \nabla^2 T, \quad (5.7)$$

where  $\mu_w$  is the viscosity of water,  $\kappa_w$  is its thermal diffusivity,  $T$  is the temperature, and  $\mathbf{q}$  and  $p$  are, as before, the velocity and pressure respectively.

Finally, the temperature  $T$  in the rock satisfies Laplace's equation

$$\nabla^2 T = 0. \quad (5.8)$$

The boundary conditions are as follows.

*In the ice,*

the velocity and pressure must satisfy an appropriate matching condition as  $y \rightarrow \infty$ ; since this condition requires a dimensionless formulation of these variables, this will be specified later.

*On the ice-water interface,  $y = S(x)$ ,*

$$(i) \text{ the traction is continuous: } S_x(p - \tau_1) + \tau_2 = p_w S_x, \quad (5.9)$$

$$S_x \tau_2 + p + \tau_1 = p_w, \quad (5.10)$$

where  $p$  represents the ice pressure, and  $p_w$  the water pressure, at the interface;

$$(ii) \text{ mass is conserved: } \rho \mathbf{q} \text{ is continuous; } \quad (5.11)$$

(iii) the rate of melting is determined by a Stefan condition:

$$\rho_I L(v_I - u_I S_x) = [k(S_x T_x - T_y)]_{\text{ice}}^{\text{water}}, \quad (5.12)$$



where  $k$  represents thermal conductivity, the suffix I refers to ice values at the interface, and the square-bracket notation is used to refer to the jump in the indicated quantity across the interface;

(iv) the temperature is at the pressure melting point:

$$T = T_M - \theta(p - p_A), \quad (5.13)$$

where  $T_M$  is the melting point at the (constant) atmospheric pressure  $p_A$ , and  $\theta$  is a constant ( $0.0074 \text{ K bar}^{-1}\dagger$ ). Note that, additionally, (5.13) holds throughout the ice region if  $x > x_M$ , and, by assumption, holds in the region  $y < y_M$  if  $x_Z < x < x_M$ . We will neglect the effect of solutes present in the water film (Hallet 1976), and that of the finite freezing rate (Nye 1973), on the melting temperature. (See § 10 for a discussion of these assumptions.)

On the water–rock interface,  $y = [y] h_R(x/[x])$ ,

$$[k \partial T / \partial n] = [T] = \mathbf{q} = 0, \quad (5.14)$$

where  $k$  and  $T$  are the conductivity and the temperature on either side of the interface and  $\partial T / \partial n$  is the normal derivative. We here suppose that the bedrock is impermeable (but see § 10).

In the rock,

$$\frac{\partial T}{\partial y} \rightarrow -\frac{G}{k_R} \quad \text{as } y \rightarrow -\infty, \quad (5.15)$$

where  $G$  is the geothermal heat flux (*ca.*  $0.05 \text{ J m}^{-2} \text{ s}^{-1}$ ), and  $k_R$  is the thermal conductivity of the bedrock.

#### Periodicity and continuity requirements

The foregoing discussion outlines the boundary conditions in  $y$  which the solution must satisfy. The boundary conditions in  $x$  are periodicity ones: particularly, since we expect (and will assume) that a solution exists in which the velocity field  $\mathbf{q}$  is twice continuously differentiable in the closure of the periodic domain  $V$  (see (6.2)) of the ice flow, we expect that the ice velocity  $\mathbf{q}$  and its first and second derivatives should be periodic. Now in terms of a stream function  $\psi$  (introduced in (6.57)), the ice-flow equations are of fourth order: hence we expect four periodicity conditions, and if we take these to be that the third derivatives of  $\psi$  are periodic, it is easy to see that  $\mathbf{q}$  and its first and second derivatives are then periodic. It follows from this that we require the ice pressure  $p$ , the stress tensor  $\sigma_{ij}$ , and indeed all the other variables in the problem to be periodic as well.

To specify the matching condition in the ice as  $y \rightarrow \infty$ , let us now consider an appropriate scaling for the model. We shall assume that there are certain scales representative of the ‘outer flow’: specifically we suppose

$$U_0, [\tau]_0, d \text{ and } l \quad (5.16)$$

are typical longitudinal velocity, stress, depth and length scales for this flow. These parameters are defined in I, where it is shown that typical values of these constants are respectively  $100 \text{ m a}^{-1}$ , 1 bar, 100 m, 10 km.

We now define

$$\nu = [y]/[x], \quad (5.17)$$

and

$$\sigma = [x]/d. \quad (5.18)$$

The parameter  $\nu$  is the bedrock-roughness slope, also considered by other authors. The parameter  $\sigma$ , not generally mentioned explicitly, may be considered to represent a measure of the bedrock roughness from the point of view of the bulk flow. Both parameters (as will be seen) are essential in obtaining an estimate of the magnitude of the basal velocity.

$\dagger \text{ bar} = 10^5 \text{ Pa.}$



We will assume that  $\sigma \ll 1, \nu < 1;$  (5.19)

we shall later find that it is an additional mathematical requirement that we consider also  $\nu \ll 1$ , to obtain physically significant sliding velocities; however, this restriction does *not* affect the method of solution. Since the dimensional basal stress (asymptotically equal to  $[\tau]_o$ ) must balance the pressure variation over the bedrock, it is natural to scale the pressure (minus its hydrostatic component) in the basal layer with  $[\tau]_o/\nu$ , and similarly for the other stress components. We therefore define the following dimensionless variables in the basal layer:

$$\left. \begin{aligned} x &= [x] \tilde{x}, & y &= [x] \tilde{y}, \\ u &= U_o \tilde{u}, & v &= U_o \tilde{v}, \\ \tau_1 &= ([\tau]_o/\nu) \tilde{\tau}_1, & \tau_2 &= ([\tau]_o/\nu) \tilde{\tau}_2, \\ p &= p_A + \rho g' d H - \rho g' [x] \tilde{y} + ([\tau]_o/\nu) \tilde{p}. \end{aligned} \right\} \quad (5.20)$$

The parameters in (5.20) have been defined already, except for  $H$ , which is the dimensionless (scaled with  $d$ ) ice thickness, measured in the  $y$ -direction; also, the stress  $[\tau]_o$  is given by its definition in I,

$$[\tau]_o = \rho g' \epsilon d. \quad (5.21)$$

The matching condition is now obtained by seeking asymptotic expansions for the outer solution in the form

$$\left. \begin{aligned} u_o &= u_o^{(0)}(x_o, y_o) + O(\sigma), \\ \tau_{2o} &= \tau_{2o}^{(0)}(x_o, y_o) + O(\sigma), \end{aligned} \right\} \quad (5.22)$$

where  $u_o$  is the longitudinal velocity and  $\tau_{2o}$  is the tangential stress. It is then shown by Fowler (1979) that by defining

$$\left. \begin{aligned} u_b &= u_b^{(0)}(x_o, h_s), \\ \tau_b &= \tau_b^{(0)}(x_o, h_s), \end{aligned} \right\} \quad (5.23)$$

the appropriate matching conditions for  $\tilde{u}$  and  $\tilde{\tau}_2$  are

$$\left. \begin{aligned} \tilde{u} &\sim u_b + O(\sigma \tilde{y}), \\ \tilde{\tau}_2 &\sim \nu [\tau_b + O(\sigma \tilde{y})], \end{aligned} \right\} \quad (5.24)$$

as  $\tilde{y} \rightarrow \infty$ . Here  $u_b$  and  $\tau_b$  are the (dimensionless) basal velocity and basal shear stress. Note that, from equations (4.8) and (4.11) of I, we can write

$$\left. \begin{aligned} \tau_b &= H [1 - \mu (H_x + h_x)], \\ \mu &= \delta/\epsilon, \quad \delta = d/l, \end{aligned} \right\} \quad (5.24a)$$

where the suffix  $x$  denotes partial differentiation with respect to a scaled (with  $l$ )  $x$ -variable,  $h$  ( $= h_s$ ) is the dimensionless bedrock,  $\mu$  is a parameter that measures the deviation of the ice surface from the mean bedrock slope, and  $\delta$  measures the shallowness of the glacier. The expression in (5.24a) follows from the scalings of I, and is a transformation to dimensionless form of Nye's classical formula  $\tau = \rho g h_1 \sin \chi$ , where  $h_1$  is the dimensional depth and  $\chi$  is the angle of inclination of the surface to the horizontal. This follows by using  $\tau = [\tau]_o \tau_b$ ,  $[\tau]_o = \rho g' \epsilon d$ ,  $g' = g/(1 + \epsilon^2)^{1/2}$ ,  $h_1 = dH$ , and  $\chi = \arctan \epsilon - \arctan [\delta (H_x + h_x)]$ . The last relation follows from the geometry of the surface, since the equation of the surface is  $y = H + h$ ; the scale factor  $\delta = d/l$  arises from the non-dimensionalization of  $x$  and  $y$  with  $l$  and  $d$  respectively. Substitution of the above relations into Nye's formula reproduces (5.24a) with an error of  $O(\delta^2)$ , which is the order of approximation to which (5.24a) is in any case valid. From I, we find that typically, if

$\arctan \epsilon = 10^\circ$ ,  $d = 100$  m,  $l = 10$  km, say, then  $\mu \sim 0.06$ . In this case, it is reasonable to neglect  $\mu$  in (5.24a) and identify the basal stress  $\tau_b$  with the depth  $H$ . Neglect of  $\mu$  in (5.24a) is equivalent to treating the ice (locally) as a slab of constant thickness.

The first condition in (5.24) is valid provided  $u_b = O(1)$ . We shall assume this latter condition to be true, since if  $u_b \sim \sigma$ , then the sliding velocity is negligible, and of little interest. *A posteriori* conditions for the validity of this assumption are given later. Note that (5.24) is the same as Nye's (1969) and Kamb's (1970) boundary condition on the ice flow at  $\infty$ , but is here placed formally in the context of an asymptotic expansion; we see how, dimensionlessly, a non-zero stress  $\nu\tau_b$  at infinity does not induce any shearing to first order in  $\sigma$  (provided  $u_b = O(1)$ ).

Applying the scalings in (5.20) to the ice flow equations, we obtain

$$\tilde{u}_{\tilde{x}} + \tilde{v}_{\tilde{y}} = 0, \quad (5.25)$$

$$\tilde{p}_{\tilde{x}} = \sigma\nu + \tilde{\tau}_{1\tilde{x}} + \tilde{\tau}_{2\tilde{y}}, \quad (5.26)$$

$$\tilde{p}_{\tilde{y}} = \tilde{\tau}_{2\tilde{x}} - \tilde{\tau}_{1\tilde{y}}, \quad (5.27)$$

where in (5.26) we have used the definition of  $[\tau]_0$  in (5.21).

The boundary conditions for large  $\tilde{y}$  may be written

$$\left. \begin{aligned} \tilde{u} &= u_b + O(\sigma\tilde{y}), \\ \tilde{v} &= 0, \\ \tilde{p} &= 0, \end{aligned} \right\} \tilde{y} \gg 1. \quad (5.28)$$

The last two conditions are obtained as follows. From I, equation (3.10), we have that the dimensionless outer pressure  $\sim \delta[\tau]_0$ ; from the scaling in (3.2) of I for the vertical velocity, and equations (3.3) and (3.4) of I, the vertical velocity  $\sim \delta U_0$ . Thus (denoting dimensionless outer variables by a suffix o),  $v_o \sim p_o \sim \delta$ . Matching then requires  $\tilde{v} \sim \tilde{p} = O(\delta)$ , but since typically  $\delta \sim 10^{-2}$ , we let  $\delta \rightarrow 0$ ; this does not affect the accuracy of our later estimates. Furthermore we only really require that the *average* value (in  $\tilde{x}$ ) of  $\tilde{p}$  should be zero as  $\tilde{y} \rightarrow \infty$ . A similar statement is true of the limiting shear stress  $\tilde{\tau}_2$  as  $\tilde{y} \rightarrow \infty$ , and we shall use these weaker conditions as required below.

Note that if  $\nu \sim 1$ , the two expressions in (5.24) are incompatible, since an  $O(1)$  stress has no corresponding form in the velocity, and we must have  $u_b \sim \sigma$ . To consider non-trivial sliding velocities  $u_b \gg \sigma$ , it is thus *formally* necessary to consider the associated limit  $\nu \rightarrow 0$ : a more precise condition is given below.

To continue our scaling, let us consider the flow in the lubrication film. We non-dimensionalize the geometry by writing

$$y = [y] [h_R(X) + \delta^* Y], \quad (5.29)$$

$$x = [x] X, \quad (5.30)$$

so that  $\delta^*$  is the dimensionless film thickness, as yet undetermined. We also define the dimensionless ice–water boundary by

$$S(x) = [y] [h_R(X) + \delta^* \Sigma(X)]. \quad (5.31)$$

For the remainder of this paper we will omit the  $\mathbf{R}$  on  $h_R$ . From (5.31),

$$S'(x) = \nu\{h' + \delta^* \Sigma'\}. \quad (5.32)$$

From (5.29) and (5.30),

$$\left. \begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{[x]} \frac{\partial}{\partial X} - \frac{h'}{\delta^*[x]} \frac{\partial}{\partial Y}, \\ \frac{\partial}{\partial y} &= \frac{1}{\delta^*[y]} \frac{\partial}{\partial Y}. \end{aligned} \right\} \quad (5.33)$$

The equation of continuity (5.5) therefore becomes

$$u_X + \frac{1}{\delta^*} \left( \frac{v}{\nu} - uh' \right)_Y = 0, \quad (5.34)$$

$u$  and  $v$  being still dimensional.

Now let us suppose that the (unknown) melting velocity given in (5.12) is written in the form

$$v_I - u_I S' = -\alpha U_0 V_M(X), \quad (5.35)$$

where the dimensionless parameter  $\alpha$  is at present unknown, but is to be chosen so that the unknown dimensionless velocity  $V_M$  is an  $O(1)$  function. For convenience, we make the inessential assumption that

$$\rho_I = \rho_w; \quad (5.36)$$

that is, the densities of ice and water are equal. Then, from (5.11),  $\mathbf{q}$  is continuous across  $S(x)$ , and the water velocity  $(u, v)$  satisfies the following equation on  $S(x)$ , where we have used (5.32):

$$v - \nu u(h' + \delta^* \Sigma') = -\alpha U_0 V_M(X). \quad (5.37)$$

We now scale  $\mathbf{q}$  in the water with a velocity  $[u]$  (to be determined), by writing

$$u = [u] U, \quad v = \nu [u] U h' + \nu \delta^* [u] V; \quad (5.38)$$

then (5.34) is

$$U_X + V_Y = 0, \quad (5.39)$$

and the boundary condition (5.37) is

$$V - U \Sigma' = -V_M(X) \quad \text{on} \quad Y = \Sigma, \quad (5.40)$$

provided we choose

$$[u] = \alpha U_0 / \nu \delta^*. \quad (5.41)$$

The other kinematic condition on  $Y = \Sigma$  may be written as

$$[u] U = U_0 \tilde{u}_I \quad \text{on} \quad Y = \Sigma, \quad (5.42)$$

where  $\tilde{u}_I$  denotes the dimensionless ice velocity in the  $x$ -direction at  $Y = \Sigma$ . By using (5.41), this is

$$U = (\nu \delta^* / \alpha) \tilde{u}_I \quad \text{on} \quad Y = \Sigma. \quad (5.43)$$

If we assume for the moment that  $\nu \delta^* / \alpha \ll 1$ , then (5.43) may be replaced by

$$U = 0 \quad \text{on} \quad Y = \Sigma, \quad (5.44)$$

in which case (5.40) is

$$V = -V_M(X) \quad \text{on} \quad Y = \Sigma. \quad (5.45)$$

The boundary condition on  $Y = 0$  is the usual no-slip one,

$$U = V = 0 \quad \text{on} \quad Y = 0. \quad (5.46)$$

In the water, it remains to nondimensionalize the temperature and pressure. For the pressure, we follow the ice scaling in (5.20), and write

$$p = p_A + \rho g' d H - \rho g' [x] \nu h + ([\tau]_0 / \nu) P; \quad (5.47)$$

guided by the Clausius–Clapeyron law (5.13), we then write

$$T = T_M - \theta[\rho g' dH - \rho g'[x] \nu h] + (\theta[\tau]_o/\nu) T^*. \quad (5.48)$$

It follows from these definitions that  $P$  and  $T^*$  are  $O(1)$ , and furthermore the Clausius–Clapeyron relation (5.13) becomes

$$T^* = -P \quad \text{on} \quad Y = \Sigma. \quad (5.49)$$

We shall use (5.48) as the scaling for the temperature in the bedrock  $\tilde{y} < h$  also.

We now anticipate that  $\delta^*$  will be so small that  $|\rho(\mathbf{q} \cdot \nabla) \mathbf{q}| \ll |\mu \nabla^2 \mathbf{q}|$  in (5.6) so that the equations reduce to those of lubrication theory (for example, Batchelor 1967). This assumption must be checked *a posteriori*. In this case, we must as usual balance the terms

$$p_x \sim \mu_w u_{yy} \quad (5.50)$$

in the first momentum equation of (5.6). From (5.47) and (5.41), this requires that we choose

$$\frac{[\tau]_o}{[x] \nu} = \frac{\mu_w \alpha U_o}{\nu \delta^{*3} [y]^2}. \quad (5.51)$$

We secondly assume that the temperature change across the film is negligible, in which case (5.49) is approximately valid on  $y = 0$ . Since the heat flux at  $Y = 0$  is continuous, we must then balance the heat absorbed by the melting ice at  $Y = \Sigma$  with that supplied from the bedrock at  $Y = 0$ ; this requires that we choose, from (5.12), (5.35) and (5.48),

$$k_R \theta[\tau]_o/\nu[x] = \rho L \alpha U_o, \quad (5.52)$$

where  $k_R$  is the thermal conductivity of the bedrock. These relations serve to determine the as yet unknown parameters  $\alpha$  and  $\delta^*$ . Multiplying (5.51) by  $k_R \theta$ , we obtain from (5.52)

$$\delta^{*3} = \mu_w k_R \theta / \rho L \nu [y]^2, \quad (5.53)$$

which gives the dimensionless film thickness. A similar result is given by Nye (1967) and Lliboutry (1968). In (5.53) we have tacitly assumed that the conductivity of ice,  $k_I$ , is less than or of order  $k_R$ ; this is generally true. We have also used the fact that the relevant length scale in the bedrock is  $[x]$ . With the values given in I, and using  $\mu_w = 10^{-2} \text{ g cm}^{-1} \text{ s}^{-1}$ , we obtain

$$\delta^* \approx 10^{-6} [x]^{\frac{1}{3}} [y]^{-1}, \quad (5.54)$$

when  $[x]$  and  $[y]$  are expressed in metres. It follows that  $\delta^* \ll 1$ , and a typical dimensional film thickness  $\delta^*[y]$  is of the order of 1  $\mu\text{m}$ .

$$\text{From (5.52) we find} \quad \alpha = k_R \theta[\tau]_o/\nu[x] \rho L U_o. \quad (5.55)$$

If we take  $U_o = 100 \text{ m a}^{-1}$ , and a typical stress  $[\tau]_o = 1 \text{ bar}$ , then, with the values of I, we find

$$\alpha \approx (2/[y]) \times 10^{-5}, \quad (5.56)$$

when  $[y]$  is expressed in metres. This implies that  $\alpha$  is negligible except for the very smallest obstacles, even if  $U_o$  is as small as 1  $\text{m a}^{-1}$ , and is the basis of the statement that regelation may be neglected in the flow except over the smallest obstacles.

Since the ice velocity satisfies, from (5.35) and (5.32),

$$\tilde{v} = \nu \tilde{u} h' - \alpha \tilde{V}_M(\tilde{x}) \quad \text{on} \quad \tilde{y} = \nu h, \quad (5.57)$$

where we have neglected terms of  $O(\delta^*)$ , it follows that the criterion for neglect of regelation is that

$$\alpha \ll \nu. \quad (5.58)$$

As stated, (5.58) is valid for all but the smallest obstacles, and it is convenient to define a transition wavelength  $\lambda_*$  (in Morland's notation), at which  $\alpha = \nu$ . From (5.55), this is given by

$$\lambda_* = k_R \theta[\tau]_0 / \nu^2 \rho L U_0. \quad (5.59)$$

This cannot be immediately compared with Morland's (1976*a*) equation (39), since  $U_0$  in (5.59) should not, strictly speaking, represent the outer velocity scale, but rather the inner velocity scale, which we are for the moment assuming is of the same order. With  $U_0 = 10 \text{ m a}^{-1}$  and  $\nu = 0.2$ , (5.59) gives  $\lambda_* = 5 \text{ mm}$ , as opposed to a value of 77 mm given by Morland. This difference is to a certain extent due to the nonlinearity of Glen's flow law for ice.

Thus generally  $\lambda_* \ll [x]$ , and the bulk flow of the ice in the basal layer may be represented as having zero normal velocity at the bedrock. Such a theoretical conclusion appears to be in conflict with certain experimental evidence, and some comments on this are made in §9. However, from our point of view, we can proceed on the basis that the above results are valid, since the physical assumptions they imply have been deduced from a rational dimensional analysis of the model.

We now justify *a posteriori* the various assumptions made in the derivation of  $\alpha$  and  $\delta^*$ . Since the temperature flux is continuous across  $Y = 0$ , one easily sees that the temperature jump across the film is of order

$$\frac{k_R [T] \delta^* [y]}{[x] k_W} \sim 4 \delta^* \nu [T] \ll [T], \quad (5.60)$$

where  $[T]$  is the typical variation of temperature in the bedrock (or ice). This justifies the assumption stated after (5.51). The assumption stated before (5.50) is justified by observing that

$$\begin{aligned} \frac{|\rho(\mathbf{q} \cdot \nabla) \mathbf{q}|}{|\mu_w \nabla^2 \mathbf{q}|} &\sim \frac{\rho [u] [y]^2 \delta^{*2}}{[x] \mu_w} \\ &\sim 10^{-10} [x]^{\frac{1}{3}} / [y] \ll 1, \end{aligned} \quad (5.61)$$

$[x]$  and  $[y]$  being measured in metres.

The final assumption, that  $\nu \delta^* / \alpha \ll 1$  (after (5.43)), is not essential, since it only affects the boundary conditions, but it does partially decouple the lubrication-layer equations from the solution for the ice flow. We have, from (5.55) and (5.54),

$$\nu \delta^* / \alpha \sim \frac{1}{2} \nu U_0 [y] \times 10^3 \times 10^{-6} [x]^{\frac{1}{3}} [y]^{-1} \sim \frac{1}{2} \nu U_0 [x]^{\frac{1}{3}} \times 10^{-3}, \quad (5.62)$$

where  $U_0$  is in metres per year, and  $[x]$  is in metres. For example, if  $U_0 = 100 \text{ m a}^{-1}$ ,  $\nu = 0.2$  and  $[x] \sim 1$ , then  $\nu \delta^* / \alpha \sim 10^{-2}$ .

We will now derive the lubrication equation for the liquid film. By neglecting inertia terms (by (5.61)), and by using (5.47), (5.33) and (5.38), the first component of the momentum equation (5.6) is

$$-\epsilon \rho g' + \left( \frac{1}{[x]} \frac{\partial}{\partial X} - \frac{h'}{\delta^* [x]} \frac{\partial}{\partial Y} \right) \left( \frac{[\tau]_0}{\nu} P - \rho g' [x] \nu h \right) = \frac{\mu_w [u]}{\delta^{*2} [y]^2} [(1 + \nu^2 h^2) U_{YY} + O(\delta^*)],$$

or, by using (5.21) and (5.51),

$$-\sigma \nu + \left( \frac{\partial}{\partial X} - \frac{h'}{\delta^*} \frac{\partial}{\partial Y} \right) \left( P - \frac{\sigma \nu^2 h}{\epsilon} \right) = (1 + \nu^2 h^2) U_{YY} + O(\delta^*). \quad (5.63)$$

The second component is

$$\rho g' + \frac{1}{\delta^* [y]} \frac{[\tau]_0}{\nu} P_Y = \frac{\mu_w \nu [u]}{\delta^{*2} [y]^2} [(1 + \nu^2 h^2) h' U_{YY} + O(\delta^*) \nu],$$

or, by using (5.21) and (5.51),

$$\frac{\sigma\nu^2}{\epsilon} + \frac{1}{\delta^*} P_Y = \nu^2[(1 + \nu^2 h'^2) h' U_{YY} + O(\delta^*)]. \quad (5.64)$$

Multiplying (5.64) by  $h'$ , and adding it to (5.63), we find, on neglecting terms of  $O(\delta^*)$ ,

$$-\sigma\nu + P_X = (1 + \nu^2 h'^2)^2 U_{YY}, \quad (5.65)$$

whereas neglecting terms of  $O(\delta^*)$  in (5.64) gives the usual independence of  $P$  on  $Y$ ,

$$P = P(X). \quad (5.66)$$

We neglect terms of  $O(\sigma)$  in (5.65), and integrate the equation twice with respect to  $Y$ , using the boundary conditions for  $U$  in (5.44) and (5.46). This gives

$$(1 + \nu^2 h'^2)^2 U = \frac{1}{2} P'(X) Y(Y - \Sigma). \quad (5.67)$$

Finally, using the equation of continuity (5.39), we have that

$$\begin{aligned} \frac{\partial}{\partial X} \int_0^\Sigma U dY &= \int_0^\Sigma U_X dY \quad (\text{since } U = 0 \text{ on } Y = \Sigma) \\ &= - \int_0^\Sigma V_Y dY = -[V]_0^\Sigma \\ &= V_M(X), \end{aligned}$$

whence

$$\int_0^\Sigma U dY = \int^X V_M(X') dX', \quad (5.68)$$

and the constant of integration is arbitrary, for the moment.

Integrating (5.67) and using (5.68), we obtain the lubrication equation for the film,

$$(1 + \nu^2 h'^2)^2 \int_{X_0}^X V_M(X') dX' = -\frac{1}{12} \Sigma^3 P'(X); \quad (5.69)$$

the constant of integration  $X_0$  is chosen so that the left side of (5.69) vanishes when  $P' = 0$ . Provided the dimensionless film thickness  $\Sigma$ , as given by (5.69), remains  $O(1)$  and positive, the complete water-film flow is given in terms of  $\Sigma$ . Thus, with this *a priori* assumption, the description of the water film uncouples from the temperature and flow problems; in other words, the regelation velocity  $V_M$  and film pressure  $P$  are determined from the solution to the temperature and ice-flow equations. However, it has been known for some time (Nye 1973; Morris 1976) that this uncoupling is not self-consistent if either  $\int_{X_0}^X V_M$  or  $P'$  becomes zero at a point  $X$  where the other is finite. In the first case, the predicted  $\Sigma$  becomes negative, in the second it becomes infinite. Morris (1979) showed that, for a particular geometry, the ice flow problem with the usual relative boundary conditions had no physically meaningful solution, and thus we are led to reconsider the formulation of the problem.

The derivation of  $\Sigma$  in (5.69) is valid so long as  $\Sigma > 0$  and  $\Sigma \sim O(1)$ : thus there is no inconsistency so long as the film is indeed thin: however if  $P' = 0$  when  $\int_{X_0}^X V_M \neq 0$ , then  $\Sigma \rightarrow \infty$ , and the derivation of (5.69) using the scalings of this section is invalid when  $\Sigma \sim O(1/\delta^*)$ . But  $\Sigma \sim O(1/\delta^*)$  corresponds exactly to situations in which the ice-water interface leaves the neighbourhood of the bedrock, and thus a cavity forms.



It is instructive to compare this notion with that of Nye (1973) in the comparable regelation theory. Nye derives an equation for the water film thickness in the form (his equation (6))

$$-T^3(dT/dS) \sin \alpha = KT^4 \cos \alpha + T^3 \cos \alpha - \psi X, \quad (5.70)$$

where

$$K = \frac{d\alpha}{dS}, \quad X = -\int_0^S \sin \alpha(S) dS, \quad (5.71)$$

$\alpha$  is the angle that the tangent to the cross section of the wire makes with the direction of the flow,  $S$  is the dimensionless distance along the circumference of the cross section, made dimensionless with a typical scale of the cross-section size, and  $T$  is the film thickness, made dimensionless in a similar manner. The symbols  $T$ ,  $X$ ,  $\alpha$ ,  $\psi$  used here in Nye's sense until equation (5.76) should not be confused with their use elsewhere in this paper. We thus have  $\alpha$ ,  $S$ ,  $d\alpha/dS \sim 1$ ;  $\psi$  is a dimensionless parameter which is easily seen, by comparison with (5.53), to be  $O(\delta^{*3})$ . Without any loss of generality, we define  $\psi = \delta^{*3}$  and put  $T = \delta^* \Sigma$ , to correspond to the scaling of the present paper. Then (5.70) is

$$-\delta^* \Sigma^3 \frac{d\Sigma}{dS} \sin \alpha = \delta^* \Sigma^4 \cos \alpha \frac{d\alpha}{dS} + \Sigma^3 \cos \alpha + \int_0^S \sin \alpha(S) dS. \quad (5.72)$$

Thus as long as  $\Sigma = O(1)$ , (5.72) gives  $\Sigma$  as

$$\Sigma^3 \cos \alpha = -\int_0^S \sin \alpha(S) dS, \quad (5.73)$$

to  $O(\delta^*)$ ; this is the analogue of (5.69). Nye's argument proceeds to show that the actual equation (5.72) has a bounded solution, since  $\Sigma$  automatically adjusts itself at points where  $\alpha = 0$  in such a way that the right side is zero; if  $X = X_0$  and  $K = K_0$  at such a point, this requires  $\Sigma$  to be a root of

$$K_0 \delta^* \Sigma^4 + \Sigma^3 = X_0 > 0, \quad (5.74)$$

where  $K_0 > 0$  if the cross section is locally convex outwards, and  $K_0 < 0$  if it is locally concave. If  $K_0 \sim O(1)$ ,  $K_0 < 0$ , then (5.74) has two positive roots,  $\Sigma \sim X_0^{1/3}$  and  $\Sigma_0 \sim 1/|K_0| \delta^*$ ; the pertinent point here is that the second is  $O(1/\delta^*)$ , and thus, if the film thickness attains this second value, Nye's corrected theory, incorporating the temperature drop across the water film, would be invalid. In fact, so long as  $X > 0$  and  $\cos \alpha > 0$  (i.e.  $\alpha \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ ), (5.73) should remain valid. This is so in regelation past irregular cross sections, until  $\alpha = \frac{1}{2}\pi$  (point B in Nye's figure 3), where if  $X = X_0 > 0$  then (5.72) implies that

$$\delta^* \Sigma^3 d\Sigma/dS = X_0, \quad (5.74a)$$

so that  $\Sigma \gg 1$  (in fact  $\Sigma \sim \delta^{*-1/2}$ ). If then  $\alpha$  increases further, we have  $\cos \alpha < 0$ ,  $d\alpha/dS > 0$ , so (5.72) is

$$\delta^* \sin \alpha \Sigma^3 \frac{d\Sigma}{dS} = X + \left[ \delta^* \Sigma^4 (-\cos \alpha) \frac{d\alpha}{dS} + \Sigma^3 (-\cos \alpha) \right], \quad (5.74b)$$

which shows that  $\Sigma$  continues to increase without limit. As long as  $1 \ll \Sigma \ll O(1/\delta^*)$ , the largest term on the right side is  $\Sigma^3(-\cos \alpha)$ ; thus for  $\alpha - \frac{1}{2}\pi \sim O(1)$ ,  $\Sigma$  increases to  $O(1/\delta^*)$  over distances  $S \sim O(1)$ . While  $\Sigma \sim O(1/\delta^*)$ , i.e.  $T \sim O(1)$ , (5.74b) is, approximately,

$$\sin \alpha \frac{dT}{dS} = -\cos \alpha \left[ 1 + T \frac{d\alpha}{dS} \right], \quad (5.75)$$

with solution

$$T \sin \alpha = -\int_{S_0}^S \cos \alpha dS. \quad (5.76)$$

If (5.76) and (5.73) are to match, we must require  $\alpha(S_0) = \frac{1}{2}\pi$ . Provided  $\alpha$  remains less than  $\pi$ , (5.76) remains valid until  $\alpha = \frac{1}{2}\pi$  again, when the thin-film approximation is resumed.

Thus, at least in principle, it seems that the formation of a cavity in regelation flow by means of 'separation' of the ice–water boundary from the ice–bedrock boundary is a serious possibility. In such conditions, the water film flow equations become invalid, and a water-filled cavity can exist. A proper formulation of the problem would then include the possibility of unknown portions of the flow where cavities existed. On the (unknown) ice–water boundary of such cavities, the water pressure would be constant, and this extra boundary condition should be sufficient to determine the cavity boundary. A similar analytic treatment of glacier sliding when cavitation occurs owing to the water pressure decreasing to the triple-point pressure has been presented by Fowler (1977), and is currently being prepared for publication.

The other possibility in (5.69) is that  $\Sigma \rightarrow 0$  when  $P' \neq 0$ . Clearly, the water film thickness cannot be negative. However, there seems no reason why  $\Sigma$  should not vanish, and one would then have a region in which the water film was absent, and an alternative boundary condition (for example, no-slip or, probably better, frictional sliding) would have to be prescribed. Such cold patches might be feasible in regelation experiments past asymmetrically cross-sectioned wires, but this is not certain, since the film thickness must be single-valued at a point whether the solution is obtained going clockwise or anti-clockwise round the wire. Since the existence of cold patches for simple regelation is not confirmed, it is consequently not clear whether their existence is possible in the sliding of fully temperate ice; note that these cold patches are not to be confused with those which must occur in *sub-temperate sliding* (Fowler 1977), which are due to the slight cooling of the ice below the melting temperature.

In the subsequent analysis we shall implicitly assume that neither regelative cavitation nor cold basal patches occur. This restricts the class of bedrock for which the analysis is self-consistent, but we justify this on the basis that we are principally interested in the nonlinearity of the flow law of ice, and its effect on the magnitude of the sliding velocity. Furthermore, the type of mathematical inconsistency that Morris (1979) describes will not be found here, since (as we show subsequently) the effect of the entire regelative mechanism can (apart from the lubrication of the bed) be neglected for the roughness scales considered.

Our point of view is that glacial sliding is a very complex phenomenon, and thus it is not realistic to try and conduct an analysis of all features of the problem. A treatment of cavitation occurring as the result of bedrock heat-exchange processes seems very difficult, except for the case of Newtonian flow over a slowly varying bedrock: it therefore seems reasonable to study other aspects of the problem separately, while bearing in mind the realistic limitations that this imposes on the results.

Let us finally describe the temperature problem in the bedrock. By using (5.48), the dimensionless temperature  $T^*$  satisfies, from (5.8),

$$\Delta T^* = 0, \quad (5.77)$$

with boundary conditions obtained from (5.49) and (5.12), by using (5.35), (5.55) and (5.60),

$$T^* = -P \quad \text{on} \quad \tilde{y} = \nu h, \quad (5.78)$$

$$[T_{\tilde{y}}^* - \nu h' T_{\tilde{x}}^*]|_{h^-} - (1/r) [T_{\tilde{y}}^* - h' T_{\tilde{x}}^*]|_{h^+} = -V_M(\tilde{x}) \quad \text{on} \quad \tilde{y} = \nu h. \quad (5.79)$$

In (5.78) and (5.79), we have used the same spatial coordinates  $(\tilde{x}, \tilde{y})$  as for the ice flow, (5.20), and we have scaled the ice temperature in the same way as for the water film, (5.48). Following

Morland's (1976 *a, b*) notation, we define the ratio,  $r$ , of ice conductivity to bedrock conductivity by

$$r = k_R/k_I. \quad (5.80)$$

Typically  $1 \lesssim r \lesssim 2$ . In (5.79), values taken on  $h^+$  and  $h^-$  refer to the values the functions take (in the limit  $\delta^* \rightarrow 0$ ) on  $\tilde{y} = \nu h$  in the ice and the bedrock respectively.

Note also that, by our assumption, we may write

$$T^* = -\tilde{p} \quad (5.81)$$

in a basal ice region near the bedrock (this being necessarily true if  $x > x_M$ ). Thus we can write (5.80) as

$$T_{\tilde{y}}^* - \nu h' T_x^* = -V_M(\tilde{x}) - r^{-1}(\tilde{p}_{\tilde{y}} - \nu h' \tilde{p}_{\tilde{x}}) \quad \text{on} \quad \tilde{y} = \nu h, \quad (5.82)$$

where  $T^*$  and  $\tilde{p}$  are understood to take values in  $h^-$  and  $h^+$  respectively. The boundary condition (5.15) is

$$\partial T^*/\partial \tilde{y} \rightarrow -\Lambda^* \quad \text{as} \quad \tilde{y} \rightarrow \infty, \quad (5.83)$$

where

$$\Lambda^* = G[x]/k_R \theta[\tau]_0. \quad (5.84)$$

If  $[x]$  is measured in metres, then typical values give  $\Lambda^* \sim 2[x]$ , which is *not* negligible (in apparent disagreement with other authors, for example, Morland 1976 *a*). However, the bedrock temperature is only important in solving the ice flow when  $\alpha \sim \nu$ , that is, for roughness scales for which  $[x] \lesssim \lambda_*$ , and when this is the case  $\Lambda^*$  is indeed negligible.

We have now completely scaled the equations and boundary conditions, except for the stress continuity conditions (5.9) and (5.10) and the flow law (5.4 *b*). The former are easily seen to be, in dimensionless form, from (5.20), (5.32) and (5.47), with  $\delta^* \rightarrow 0$ ,

$$\tilde{\tau}_2 + \nu h'(\tilde{p} - \tilde{\tau}_1) = \nu P h', \quad (5.85)$$

$$\tilde{p} + \tilde{\tau}_1 + \nu h' \tilde{\tau}_2 = P, \quad (5.86)$$

both on  $\tilde{y} = \nu h$ . Let us define, from (5.4 *b*),

$$e = (U_0/2[x]) \tilde{e}, \quad \tau = ([\tau]_0/\nu) \tilde{\tau} \quad (5.87)$$

( $\tilde{e}_{ij}$  and  $\tilde{\tau}_{ij}$  are defined equivalently), so that (see I, equation (3.6))

$$\tilde{e} = [(\tilde{u}_{\tilde{y}} + \tilde{v}_{\tilde{x}})^2 + 4\tilde{u}_{\tilde{x}}^2]^{\frac{1}{2}}, \quad (5.88)$$

$$\tilde{\tau} = (\tilde{\tau}_1^2 + \tilde{\tau}_2^2)^{\frac{1}{2}}. \quad (5.89)$$

Then we have, from (5.4 *b*),

$$e = A\tau^n,$$

whence

$$\frac{U_0}{2[x]} \tilde{e} = A \frac{[\tau]_0^n}{\nu^n} \tilde{\tau}^n. \quad (5.90)$$

Also, from the definition of  $[\tau]_0$  in I,

$$[\tau]_0^n = U_0/2Ad. \quad (5.91)$$

Eliminating  $A$  from (5.90) and (5.91), we obtain, using (5.18), the dimensionless flow law relating  $\tilde{e}$  and  $\tilde{\tau}$ , defined by (5.88) and (5.89):

$$\tilde{e} = (\sigma/\nu^n) \tilde{\tau}^n. \quad (5.92)$$

For completeness, we here set out the complete coupled bedrock-ice flow problems to be solved. For convenience, we will omit the tildes on dimensionless variables, and the asterisk on  $T^*$ .

In  $y > \nu h$ ,

$$\left. \begin{aligned} e_{ij} &= \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}, \\ e_{ij} &= (\sigma/\nu^n) \tau^{n-1} \tau_{ij}, \quad e = (\sigma/\nu^n) \tau^n, \\ e &= [(u_y + v_x)^2 + 4u_x^2]^{\frac{1}{2}}, \quad \tau = [\tau_1^2 + \tau_2^2]^{\frac{1}{2}}, \\ u_x + v_y &= 0, \\ \dot{p}_x &= \sigma\nu + \tau_{1x} + \tau_{2y}, \\ \dot{p}_y &= \tau_{2x} - \tau_{1y}. \end{aligned} \right\} \quad (5.93)$$

In  $y < \nu h$ ,

$$\Delta T = 0. \quad (5.94)$$

As  $y \rightarrow \infty$  (or  $y \gg 1$ ),

$$\left. \begin{aligned} u &\sim u_b + O(\sigma y), \\ v, \dot{p} &\rightarrow 0, \\ \tau_2 &\sim \nu[\tau_b + O(\sigma y)]. \end{aligned} \right\} \quad (5.95)$$

(It is sufficient to take the average values of  $\tau_2$  and  $\dot{p}$  in (5.95).)

On  $y = \nu h$ ,

$$\begin{aligned} v &= \nu h' - \alpha V_M(x), \\ (1 - \nu^2 h'^2) \tau_2 - 2\nu h' \tau_1 &= 0 \end{aligned}$$

(by eliminating  $P$  from (5.85) and (5.86)),

$$\left. \begin{aligned} T &= -\dot{p}, \\ T_y - \nu h' T_x &= -V_M(x) - r^{-1}(\dot{p}_y - \nu h' \dot{p}_x). \end{aligned} \right\} \quad (5.96)$$

As  $y \rightarrow -\infty$ ,

$$\partial T / \partial y \rightarrow -\Lambda^*. \quad (5.97)$$

The system (5.93)–(5.97) is to be solved to find  $\tau_b$  as a function of  $u_b$ . It is based on the main assumptions that (i) the ice flow is independent of the moisture content; and (ii) either  $x > x_M$ , or there exists a basal region next to the bedrock where the ice is temperate.

If we suppose that  $h$  has period  $M$ , then we further constrain our solutions to be periodic in  $x$  with the same period.

## 6. A VARIATIONAL PRINCIPLE FOR THE ICE FLOW

If we examine the system (5.93)–(5.97), we see that it is dependent on five dimensionless parameters:  $\sigma$ ,  $\nu$ ,  $\alpha$ ,  $r$  and  $\Lambda^*$ . Of these,  $\sigma$  and  $\nu$  occur in the ice-flow equations;  $\nu$ ,  $\alpha$  and  $r$  appear in the boundary conditions on the ice–rock interface; and  $\Lambda^*$  represents the geothermal heat flux to the bedrock. The parameters  $\sigma$  and  $\nu$  are crucial in determining the magnitude of the basal velocity  $u_b$ , as should be clear from (5.93). The parameters  $r^{-1} \lesssim 1$  and  $\Lambda^* \sim 1$  are inessential:  $r^{-1}$  merely reinforces the bedrock thermal gradient (since, for example, upstream,  $\partial T / \partial n$  and  $\partial \dot{p} / \partial n$  are both positive); it is unlikely to affect substantially the scale or effects of regelation, although it will increase  $V_M$ . Similarly, although  $\Lambda^* \sim 1$ , the rock-temperature problem only becomes of interest when regelation cannot be neglected; as already stated, this occurs over bedrock roughness on the scale of the transition wavelength  $\lambda_* \ll [x]$ , and over such scales the effect of  $\Lambda^*$  will indeed be negligible.

The remaining parameter,  $\alpha$ , is the crucial one in determining a solution of (5.93)–(5.97). It represents the effects of regelation. We have shown that  $\alpha \ll 1$ , so that (provided  $V_M \lesssim 1$ ) it seems reasonable to neglect  $\alpha$  in (5.96). The ice flow problem then uncouples altogether from the bedrock temperature field. However, we can only guarantee  $V_M \lesssim 1$  when the relevant length scale is  $[\lambda]$ . Over much smaller length scales,  $\partial T/\partial n$  increases and  $V_M$  may become so large that  $\alpha V_M \sim 1$ . Thus, neglect of  $\alpha$  corresponds precisely to assuming that roughness is absent at wavelengths less than about  $\lambda_*$ ; such an assumption corresponds to Kamb's (1970) 'truncated white bedrock', and may often be valid. For the remainder of this section, we therefore assume that

$$\alpha = 0. \quad (6.1)$$

The effect of a small, but non-zero,  $\alpha$  is briefly discussed in §8.

We now consider a variational principle for the ice flow problem, (5.93), (5.95) and (5.96), with  $\alpha = 0$ . Such a principle was first comprehensively put forward by Johnson (1960, 1961). For certain stress–strain rate relations he stated a general variational principle that has as its Euler equations and natural boundary conditions the equations and boundary conditions of the steady, slow flow of a non-Newtonian fluid bounded by a surface on which appropriate velocity and stress conditions are given. Specifically, he considered the bounding surface  $S$  to be composed of non-overlapping components  $S_t$  and  $S_v$  on which, respectively, the stress and velocity components were specified. In this case, by also using natural admissibility conditions, velocity and stress principles may be deduced. For certain flow laws (of which the power-law model is one) these give a global maximum and minimum for the variational functional. For the power-law model, this functional is just a multiple of the drag. Hence, by finding appropriate trial functions, we can estimate the drag on the bedrock, i.e. the basal stress.

Johnson's theory has been widely applied. Wasserman & Slattery (1964) used it to estimate the drag on a sphere moving slowly in an unbounded fluid, and, later, Hopke & Slattery (1970) obtained bounds on the drag on a sphere moving slowly in an Ellis-model fluid, which is similar to a power-law fluid at large stresses, but which has finite viscosity at small stresses. This suggests itself as a useful model for the rheology of ice (cf. Budd & Radok 1971).

No correctly formulated variational principle for a problem, such as flow past a bubble, that requires mixed boundary conditions (no normal velocity, no tangential stress) seems to have been yet considered. A slight modification of Johnson's principle is necessary, by means of a method well known in linear elasticity.

It is not surprising that all the work to date has been done on spherical geometries, since the well known Stokes paradox (for example, Proudman & Pearson 1957) does not appear explicitly in this case. Thus one can obtain a meaningful estimate for the drag even when the inertial terms are neglected everywhere in the flow; the same would not be true in a cylindrical geometry. The only work on non-Newtonian fluid flow that explicitly incorporates a description of the outer (Oseen) flow away from the inner (Stokes) flow is that by Caswell & Schwarz (1962) for a Rivlin–Ericksen fluid.

To formulate properly a variational principle for a slow flow in an infinite expanse of fluid, it is necessary to solve the outer flow problem to first order, obtain the appropriate matching conditions for the inner flow, and prescribe these conditions on a 'boundary' in the matching region. This 'asymptotic variational principle' will give estimates for the solution that are accurate to the same order as the matching conditions. More formally, it provides estimates for the leading-order term in an asymptotic expansion of the inner flow. If complementary varia-



tional principles exist, then these provide upper and lower bounds for the variational functional to first order.

This is essentially how the bounding surface is chosen in the problem considered here. In this case, however, the whole glacier is well within the region of Stokes flow since the expanse of fluid is bounded, and the Reynolds number is so small (*ca.*  $10^{-13}$ ) that the inertia terms remain negligible throughout the flow. The inner and outer regions are simply the basal and main-stream regions of the flow, corresponding to expansions of the flow solutions in powers of  $\sigma$ . With the dimensionless coordinates introduced in §5, we therefore choose the volume  $V$  of the basal flow to be

$$V = (0 < x < M, \nu h(x) < y < y^*), \quad (6.2)$$

where  $M$  is the period of  $h$ , and  $1 \ll y^* \ll 1/\sigma$ , (6.3)

so that  $y = y^*$  lies in the matching region between inner and outer flows (Cole 1968). The bounding surface  $S$  of the flow is therefore just the boundary of  $V$ ; the geometry is shown in figure 3.

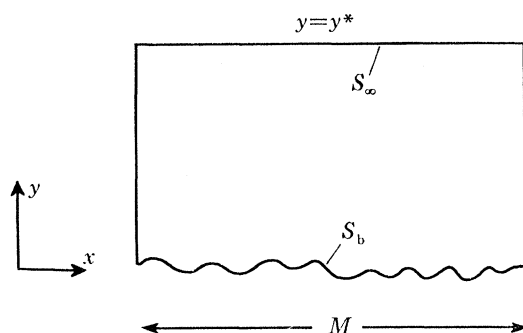


FIGURE 3. Bounding surface geometry for application of the variational principle.

We will denote the bedrock portion of  $S$ ,  $y = \nu h$ , as  $S_b$ , and the matching region part,  $y = y^*$ , as  $S_\infty$ .

We may write the dimensionless equations of ice flow given in §5, with  $\alpha = 0$ , as

$$v_{i,i} = 0, \quad (6.4)$$

$$\sigma_{ij,j} + f_i = 0, \quad (6.5)$$

where

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \quad (6.6)$$

and

$$\mathbf{f} = (\sigma\nu, 0). \quad (6.7)$$

The rate-of-deformation tensor is

$$e_{ij} = v_{i,j} + v_{j,i}, \quad (6.8)$$

and we shall suppose that there is a function  $\Gamma(e_{rs})$  such that the flow law given in (5.93) may be written in the form

$$\tau_{ij} = 2 \partial\Gamma/\partial e_{ij} = \tau_{ji}. \quad (6.9)$$

Equations (6.4)–(6.9) are to be solved subject to the conditions that the solutions be periodic in  $x$  with period  $M$ ,  $v_i$  is twice continuously differentiable on the closure of  $V$ , and, on  $S_\infty$ ,

$$\left. \begin{aligned} u &= u_b + O(\sigma y^*), \quad v = 0, \\ \int_0^M p \, dx &= 0, \\ \frac{1}{M} \int_0^M \tau_2 \, dx &= \nu[\tau_b + O(\sigma y^*)]; \end{aligned} \right\} \quad (6.10)$$



$$\text{on } S_b, \quad \left. \begin{aligned} v_i n_i &= v_n = 0, \\ \sigma_{ij} n_i t_j &= \sigma_{nt} = -\tau_s(x). \end{aligned} \right\} \quad (6.11)$$

In these equations,  $\mathbf{v} = (v_1, v_2)$  is the velocity, suffixes  $i$  and  $j$  are used to denote components in cartesian geometry,  $\mathbf{n}$  and  $\mathbf{t}$  denote unit (outward) normal and tangential vectors to the bedrock  $S_b$ . Commas denote partial differentiation with respect to the indicated coordinate, and we employ the summation convention. If there is no tangential stress on  $S_b$ , then the traction  $\tau_s = 0$ , but we retain  $\tau_s$  in (6.11) in case we wish to model such effects as basal friction due to debris (Morland 1976*b*). If a basal friction is applied, then  $\tau_s > 0$  in (6.11).

Let us consider the functional

$$\begin{aligned} J = \int_V & \left[ \frac{1}{2}(v_{i,j} + v_{j,i} - e_{ij}) \tau_{ij} + \Gamma - f_i v_i - p v_{i,i} \right] dV \\ & - \int_{S_b} (\sigma_{kj} n_k n_j v_i n_i - \tau_s v_i t_i) dS \\ & - \int_{S_\infty} \sigma_{ij} n_j (v_i - U_i) dS, \end{aligned} \quad (6.12)$$

$$\text{where} \quad \mathbf{U} = (U_1, U_2) = (u_b + O(\sigma y^*), 0). \quad (6.13)$$

This is the equivalent of the functional considered by Johnson (1961), except that only the components corresponding to the terms that are known in the surface integral over  $S_b$  are included. The second integrand may be written more briefly as

$$\sigma_{nn} v_n - \tau_s v_t. \quad (6.14)$$

Now let  $v_i, \tau_{ij}, \sigma_{ij}, p, e_{ij}$  be a solution of equations (6.4), (6.5), (6.6), (6.8) and (6.9), with  $\mathbf{f}$  given by (6.7), that satisfies the periodicity and differentiability requirements together with the boundary conditions (6.10) and (6.11). Also, let  $\delta v_i$  and  $\delta \tau_{ij}$ , etc., be arbitrary variations of these functions such that  $\delta v_i$  is continuous, piecewise continuously differentiable and periodic, but  $\delta \sigma_{ij}$  need be neither continuous nor periodic. Denoting the first-order variation of the functional by  $\delta J$ , we then find that

$$\begin{aligned} \delta J = \int_V & \left[ \frac{1}{2}(v_{i,j} + v_{j,i} - e_{ij}) \delta \tau_{ij} + \left( \frac{\partial \Gamma}{\partial e_{ij}} - \frac{1}{2} \tau_{ij} \right) \delta e_{ij} + \tau_{ij} \delta v_{i,j} - f_i \delta v_i - p \delta v_{i,i} - \delta p v_{i,i} \right] dV \\ & - \int_{S_b} [v_i n_i \delta \sigma_{kj} n_k n_j + (\sigma_{kj} n_k n_j n_i - \tau_s t_i) \delta v_i] dS \\ & - \int_{S_\infty} [(v_i - U_i) \delta \sigma_{ij} n_j + \sigma_{ij} n_j \delta v_i] dS, \end{aligned} \quad (6.15)$$

where we have used the fact that  $\tau_{ij} = \tau_{ji}$ .

Using the equations (6.9), (6.8), (6.6) and (6.4), we may rewrite the volume integral in (6.15) as

$$\int_V [(\sigma_{ij} \delta v_{i,j})_{,j} - \{\sigma_{ij,j} + f_i\} \delta v_i] dV, \quad (6.16)$$

and then using (6.5), periodicity, and Green's theorem on (6.16), we obtain from (6.15)

$$\delta J = - \int_{S_b} [v_i n_i \delta \sigma_{kj} n_k n_j + (\sigma_{kj} n_k n_j n_i - \tau_s t_i) \delta v_i] dS + \int_{S_b} \sigma_{ij} n_j \delta v_i dS - \int_{S_\infty} (v_i - U_i) \delta \sigma_{ij} n_j dS. \quad (6.17)$$

Using the boundary conditions in (6.11), together with (6.10) and (6.13), we obtain

$$\delta J = \int_{S_b} (\sigma_{ij} n_j - \sigma_{kj} n_k n_j n_i - \sigma_{ij} n_i t_j t_i) \delta v_i dS. \quad (6.18)$$

Now,

$$\begin{aligned} \sigma_{kj} n_k n_j n_i + \sigma_{ij} n_i t_j t_i &= \sigma_{nn} n_i + \sigma_{nt} t_i \\ &= \sigma_{in} \\ &= \sigma_{ij} n_j, \end{aligned} \quad (6.19)$$

by the usual rules of tensor manipulation. Therefore, the integrand in (6.18) is zero, and so

$$\delta J = 0; \quad (6.20)$$

thus  $J$  is stationary at a solution of the problem.

If, instead of (6.9), we can write the flow law in the form

$$e_{ij} = e_{ji} = 2(\partial \hat{\Gamma}(\tau_{rs}) / \partial \tau_{ij}), \quad (6.21)$$

and then define a functional  $\mathcal{H}$  by

$$\mathcal{H} = J + \int_V [\frac{1}{2} e_{ij} \tau_{ij} - \Gamma(e_{rs}) - \hat{\Gamma}(\tau_{rs})] dV, \quad (6.22)$$

it is easy to see that also

$$\delta \mathcal{H} = 0 \quad (6.23)$$

at a solution of the boundary value problem; furthermore (as is shown below), for a power-law fluid we have

$$\Gamma(e_{rs}) + \hat{\Gamma}(\tau_{rs}) = \frac{1}{2} e_{ij} \tau_{ij} \quad (6.24)$$

at a solution, and therefore, in view of (6.22),

$$\mathcal{H} = J = J_0, \quad (6.25)$$

say, at such a solution, and from (6.12), the equations of motion (6.4) and (6.8), and the boundary conditions (6.10) and (6.11),

$$J_0 = \int_V (\Gamma - f_i v_i) dV + \int_{S_b} \tau_s v_t dS. \quad (6.26)$$

We will now show that the power law described by (5.93) may be written in the form (6.9) or (6.21). (Application of the variational principles for a more general class of flow law, such as an Ellis model or polynomial law, is discussed in Appendix B.) We have

$$e_{ij} e_{ij} = 2e^2, \quad \tau_{ij} \tau_{ij} = 2\tau^2. \quad (6.27)$$

Let us define

$$\Gamma(e_{rs}) = m e^{(n+1)/n}, \quad \hat{\Gamma}(\tau_{rs}) = \hat{m} \tau^{n+1}. \quad (6.28)$$

From (6.27), we have

$$2e \frac{\partial e}{\partial e_{ij}} = e_{ij}, \quad 2\tau \frac{\partial \tau}{\partial \tau_{ij}} = \tau_{ij}, \quad (6.29)$$

and therefore

$$\frac{\partial \Gamma}{\partial e_{ij}} = \frac{n+1}{n} m e^{1/n} \frac{\partial e}{\partial e_{ij}} = \frac{1}{2} \frac{n+1}{n} m e^{-(n-1)/n} e_{ij}. \quad (6.30)$$

From (5.93), we can write the flow law in the form

$$\tau_{ij} = \frac{\tau}{e} e_{ij} = \left( \frac{\sigma}{\nu^n} \right)^{-1/n} e^{-(n-1)/n} e_{ij}, \quad (6.31)$$

which is of the form (6.9) if, from (6.30),

$$m = \frac{n}{n+1} \left( \frac{\sigma}{\nu^n} \right)^{-1/n}. \quad (6.32)$$

Similarly, from (6.29) and (6.28),

$$\begin{aligned}\frac{\partial \hat{F}}{\partial \tau_{ij}} &= (n+1) \hat{m} \tau^n \frac{1}{2\tau} \tau_{ij} \\ &= \frac{1}{2}(n+1) \hat{m} \tau^{n-1} \tau_{ij},\end{aligned}\quad (6.33)$$

and so  $e_{ij}$  is of the form (6.21), provided we choose, from (5.93),

$$\hat{m} = \frac{1}{n+1} \frac{\sigma}{\nu^n}. \quad (6.34)$$

We therefore define  $\Gamma$  and  $\hat{F}$  by (6.28), with  $m$  and  $\hat{m}$  defined by (6.32) and (6.34). We can now evaluate the volume integral in (6.26). From (6.27) and (5.93) we find that

$$\frac{1}{2} \tau_{ij} e_{ij} = \frac{\sigma}{\nu^n} \tau^{n+1} = \left(\frac{\sigma}{\nu^n}\right)^{-1/n} e^{(n+1)/n}. \quad (6.35)$$

Comparing (6.35) with (6.28), (6.32) and (6.34), we find that

$$\frac{1}{2} \tau_{ij} e_{ij} = (n+1) \hat{F} = [(n+1)/n] \Gamma \quad (6.36)$$

at a solution. This confirms the relation (6.24), and shows that

$$\begin{aligned}\int_V \Gamma \, dV &= \frac{n}{n+1} \int_V \frac{1}{2} \tau_{ij} e_{ij} \, dV \\ &= \frac{n}{n+1} \int_V \sigma_{ij} v_{i,j} \, dV,\end{aligned}$$

by using (6.8), (6.6) and (6.4), and therefore, from (6.7) and (6.5),

$$\begin{aligned}\int_V (\Gamma - f_i v_i) \, dV &= \frac{n}{n+1} \int_V \left[ -\sigma_{ij,j} v_i + (\sigma_{ij} v_i)_{,j} - \frac{n+1}{n} f_i v_i \right] \, dV \\ &= \frac{n}{n+1} \left( \int_{S_\infty} \sigma_{ij} v_i n_j \, dS + \int_{S_b} \sigma_{ij} v_i n_j \, dS \right) - \frac{1}{n+1} \int_V \sigma \nu \, dV.\end{aligned}\quad (6.37)$$

We note that on  $S_b$

$$\begin{aligned}\sigma_{ij} v_i n_j &= \sigma_{in} v_i = \sigma_{nn} n_i v_i + \sigma_{nt} t_i v_i \\ &= -\tau_s v_t,\end{aligned}\quad (6.38)$$

using (6.11), and therefore from the boundary conditions (6.10) (or 5.95) we have, using the fact that  $u \lesssim u_b + O(\sigma y^*)$  in  $V$ ,

$$\begin{aligned}J_0 &= \int_V (\Gamma - f_i v_i) \, dV + \int_{S_b} \tau_s v_t \, dS \\ &= \frac{n}{n+1} M\nu [\tau_b + O(\sigma y^*)] [u_b + O(\sigma y^*)] + \frac{1}{n+1} \int_{S_b} \tau_s v_t \, dS + O\left(\frac{\sigma \nu u_b M y^*}{n+1}\right) \\ &= \frac{1}{n+1} \int_{S_b} \tau_s v_t \, dS + \frac{n}{n+1} M\nu \tau_b u_b + O(\sigma y^*)\end{aligned}\quad (6.39)$$

as  $\sigma \rightarrow 0$ .

Now let us construct complementary variational principles for the given functional  $J_0$ , using the functionals  $J$  and  $\mathcal{H}$ . Firstly, we construct a *velocity principle* by considering variations of  $J$ , when the admissible functions are restricted so that equations (6.4), (6.8) and (6.9), the first two

conditions in (6.10) and the first condition in (6.11) remain satisfied under the variation. Defining the functional  $J$  with these restrictions to be  $J_V$ , we immediately see from (6.12) that

$$J_V = \int_V (\Gamma - f_i v_i) dV + \int_{S_b} \tau_s v_t dS. \quad (6.40)$$

The first variation  $\delta J_V$  is zero at a solution (at which also  $J_V = J_0$ ), and the remainder can be written

$$J_2 = \int_V \left[ \Gamma(e_{ij} + \delta e_{ij}) - \Gamma(e_{ij}) - \frac{\partial \Gamma}{\partial e_{ij}} \delta e_{ij} \right] dV. \quad (6.41)$$

It is shown by Johnson (1961) that the integrand in (6.41) is always positive, and therefore (since  $\delta J_V = 0$ ) we can write

$$J_V = J_0 + J_2 \geq J_0, \quad (6.42)$$

so that  $J_0$  as defined by (6.39) provides a global minimum for the functional  $J_V$  in (6.40), when  $e_{ij}$  and  $v_i$  satisfy the afore-mentioned conditions.

In a similar manner, we define a *stress principle* by considering variations of  $\mathcal{H}$ , and restricting admissible variations so that (6.21), (6.5), (6.6), the third and fourth boundary conditions in (6.10) and the second one in (6.11) are satisfied. Defining this restricted functional to be  $\mathcal{H}_\tau$ , we find, from (6.12), replacing  $\Gamma - \frac{1}{2}e_{ij}\tau_{ij}$  by  $\hat{\Gamma}$ , that

$$\mathcal{H}_\tau = \int_V -\hat{\Gamma} dV + \int_{S_\infty} \sigma_{ij} U_i n_j dS. \quad (6.43)$$

We know that  $\mathcal{H}_\tau = J_0$  at a solution of the boundary value problem, and also that  $\delta \mathcal{H}_\tau = 0$ . The remainder,  $\mathcal{H}_2$ , is given from (6.43) by

$$\mathcal{H}_2 = - \int_V \left[ \hat{\Gamma}(\tau_{ij} + \delta \tau_{ij}) - \hat{\Gamma}(\tau_{ij}) - \frac{\partial \hat{\Gamma}}{\partial \tau_{ij}} \delta \tau_{ij} \right] dV, \quad (6.44)$$

and is less than zero in precisely the same manner that  $J_2$  is greater than zero. It follows that we may write

$$\mathcal{H}_\tau = J_0 + \mathcal{H}_2 \leq J_0. \quad (6.45)$$

The inequalities in (6.42) and (6.45) are strict unless the variations are zero. These results incidentally prove that there can be at most one solution of the problem.

We have obtained global bounds on  $J_0$ ,

$$\mathcal{H}_\tau \leq J_0 \leq J_V, \quad (6.46)$$

and can now use (6.46) and the definition of  $J_0$  in (6.39) to obtain estimates for the basal stress  $\tau_b$ .

With realistic estimates, we can neglect  $f_i v_i$  in (6.40), and then we find that the asymptotic complementary variational principles give, to first order, on letting  $\sigma \rightarrow 0$ ,

$$\begin{aligned} u_b \int_{S_\infty} \tau_2 dS - \frac{1}{n+1} \frac{\sigma}{\nu^n} \int_V \tau^{n+1} dV &\leq \frac{1}{n+1} \int_{S_b} \tau_s v_t^s dS + \frac{n}{n+1} M \nu \tau_b u_b \\ &\leq \frac{n}{n+1} \left( \frac{\sigma}{\nu^n} \right)^{-1/n} \int_V e^{(n+1)/n} dV + \int_{S_b} \tau_s v_t dS, \end{aligned} \quad (6.47)$$

where we have used (6.35) and (6.36), and  $v_t^s$  denotes the solution  $v_t$  and not a trial function.

At this point note that we have not assumed, in deriving (6.47), that  $\nu \ll 1$ . Thus these estimates would seem to be applicable to bedrocks with non-vanishing mean slope. However, our main assumption in deriving the first-order bounds above is that  $u_b \gg O(\sigma)$ ; if  $u_b \lesssim O(\sigma)$ , then,

from the point of view of the large-scale ice dynamics, we could reasonably take  $u \approx 0$  (cf. Richardson 1973). Now we can at once establish the magnitude of  $u_b$  by noting that, at the (first-order) solution of the problem, the inequalities in (6.47) becomes equalities. Let us suppose that

$$\tau_s = 0 \quad (6.48)$$

(no tangential stress at the bedrock). Then, since  $\tau_2 = \nu\tau_b$  on  $S_\infty$ , we have, at the solution,

$$\nu\tau_b u_b M = \frac{\sigma}{\nu^n} \int_0^M \int_{\nu h}^{y^*} \tau^{n+1} dV, \quad (6.49)$$

$$u_b \sim \sigma/\nu^{n+1}, \quad (6.50)$$

and thus, *formally*, we do indeed require

$$\nu \ll 1 \quad (6.51)$$

for non-negligible sliding velocities. This assumption is that taken for granted by other authors. We stress the formality of this result: if  $n \approx 3$  and  $\sigma \sim 10^{-2}$  (for example, a roughness wavelength of 1 m beneath 100 m of ice), then  $u_b \sim 1$  if  $\nu \sim \sigma^{1/(n+1)} \sim 0.3$ , which is hardly infinitesimal. However, this is of little importance in (6.47), since no such approximation as (6.51) has been made there. We note that, as  $\sigma \rightarrow 0$ ,  $y^* \rightarrow \infty$ , but the right side of (6.49) does not converge since  $\tau \rightarrow \nu\tau_b$  as  $y^* \rightarrow \infty$ . This is easily remedied by writing

$$\begin{aligned} \frac{\sigma}{\nu^n} \int_V \tau^{n+1} dV &= \frac{\sigma}{\nu^n} \int (\tau^{n+1} - \nu^{n+1} \tau_b^{n+1}) dV + O(M\sigma\nu y^*) \\ &\approx \frac{\sigma}{\nu^n} \int_V (\tau^{n+1} - \nu^{n+1} \tau_b^{n+1}) dV \end{aligned} \quad (6.52)$$

to first order as  $\sigma \rightarrow 0$ . With this adjustment we can extend the  $y$ -integral in (6.49) to  $y^* = \infty$ . Defining

$$u_b = (\sigma/\nu^{n+1}) u_b^*, \quad (6.53)$$

so that  $u_b^* = O(1)$ , we have at a solution, from the second equality in (6.47) with  $\tau_s = 0$ ,

$$M\tau_b \frac{\sigma}{\nu^n} u_b^* = \left(\frac{\sigma}{\nu^n}\right)^{-1/n} \int_V e^{(n+1)/n} dV \quad (6.54)$$

and it follows from (6.54) that  $e$  is not scaled to be  $O(1)$  in (5.93), but that

$$e = O(\sigma/\nu^n), \quad (6.55)$$

(as is also obvious from (5.92)).

Now let us define the Airy stress function which satisfies

$$\left. \begin{aligned} \sigma_{11} &= -p + \tau_1 = \phi_{yy}, \\ \sigma_{12} &= \sigma_{21} = \tau_2 = -\phi_{xy}, \\ \sigma_{22} &= -p - \tau_1 = \phi_{xx}, \end{aligned} \right\} \quad (6.56)$$

when (6.5) is satisfied (neglecting  $f_i$ ). Also when (6.4) is satisfied, there exists a stream function  $\psi$  satisfying

$$u = \psi_y, \quad v = -\psi_x. \quad (6.57)$$

Motivated by (6.53), (6.55) and the magnitude of the basal stress, we now rescale the dimensionless functions  $\psi$  and  $\phi$  by defining

$$\psi = \nu u_b \psi^* = (\sigma/\nu^n) u_b^* \psi^*, \quad (6.58)$$

$$\phi = \tau_b \phi^*. \quad (6.59)$$

Thus (6.55) is automatically satisfied, and we find that the inequalities in (6.47) become (still with  $\tau_s = 0$ ), using (6.52),

$$\begin{aligned} & \frac{\sigma}{\nu^{n+1}} u_b^* \tau_b [-\phi_y^*]_0^M - \frac{1}{n+1} \frac{\sigma}{\nu^n} \tau_b^{n+1} \int_V (\tau^{*n+1} - \nu^{n+1}) dV \\ & \leq \frac{n}{n+1} M \frac{\sigma}{\nu^n} \tau_b u_b^* \leq \frac{n}{n+1} \frac{\sigma}{\nu^n} u_b^{*(n+1)/n} \int_V e^{*(n+1)/n} dV, \end{aligned} \quad (6.60)$$

where  $e^*$  and  $\tau^*$  are defined by

$$e^* = [(\psi_{yy}^* - \psi_{xx}^*)^2 + 4\psi_{xy}^{*2}]^{\frac{1}{2}}, \quad (6.61)$$

$$\tau^* = [\phi_{xy}^{*2} + \frac{1}{4}(\phi_{yy}^* - \phi_{xx}^*)^2]^{\frac{1}{2}}. \quad (6.62)$$

In (6.60) we require only that  $\psi^*$  be continuously differentiable and piecewise twice continuously differentiable,

$$\psi^* \sim y/\nu \quad \text{as } y \rightarrow \infty, \quad (6.63)$$

$$\psi^* = 0 \quad \text{on } y = \nu h, \quad (6.64)$$

as conditions on  $\psi^*$ , and 
$$\int_0^M (\phi_{yy}^* + \phi_{xx}^*) dx \rightarrow 0 \quad \text{as } y \rightarrow \infty, \quad (6.65)$$

and, from (6.11), (5.96) and (6.56),

$$(1 - \nu^2 h'^2) \phi_{xy}^* + \nu h'(\phi_{yy}^* - \phi_{xx}^*) = 0 \quad \text{on } y = \nu h. \quad (6.66)$$

Since we have  $e^* \sim 1$ , it follows from (6.63) that we may write the stream function in the form

$$\psi^* = y/\nu + \psi_1, \quad \psi_1 = O(1), \quad (6.67)$$

so that (if  $\nu \ll 1$ ), the basal velocity is unaffected, to first order in  $\nu$ , by the bedrock.

The shear stress condition at  $y = \infty$  may be written as

$$[-\phi_y^*]_0^M = M\nu, \quad (6.68)$$

and so we suppose that  $\phi^*$  satisfies

$$\phi_{xy}^* \sim -\nu\alpha(x), \quad y \rightarrow \infty, \quad (6.69)$$

where  $\alpha(x)$  is to be chosen such that it is periodic in  $x$  with period  $M$  and mean equal to one, i.e.

$$\frac{1}{M} \int_0^M \alpha(x) dx = 1. \quad (6.70)$$

In this case (6.68) and (6.65) are both satisfied, and the bounds in (6.60) become

$$\begin{aligned} u_b^* \tau_b - \frac{1}{n+1} \tau_b^{n+1} \frac{1}{M} \int_V (\tau^{*n+1} - \nu^{n+1}) dV & \leq \frac{n}{n+1} \tau_b u_b^* \\ & \leq \frac{n}{n+1} u_b^{*(n+1)/n} \frac{1}{M} \int_V e^{*(n+1)/n} dV, \end{aligned} \quad (6.71)$$

whence we obtain

$$\tau_b \leq u_b^{*1/n} \frac{1}{M} \int_V e^{*(n+1)/n} dV, \quad (6.72)$$

$$u_b^* \leq \tau_b^n \frac{1}{M} \int_V (\tau^{*n+1} - \nu^{n+1}) dV, \quad (6.73)$$

where  $e^*$  and  $\tau^*$  are defined by (6.61) and (6.62), and  $\psi^*$  and  $\phi^*$  satisfy the constraints (6.69), (6.63), (6.64) and (6.66). (6.72) and (6.73) are valid to first order in  $\sigma$ , provided we have  $\nu \ll 1$  (so that  $\sigma/\nu^{n+1} \gg \sigma$ ). We emphasize that in practice the large value of  $n$  means that  $\nu$  may be



considered 'small' for  $\nu \lesssim \frac{1}{2}$ .  $e^*$  and  $\tau^*$  are  $O(1)$  quantities, and furthermore, the boundary conditions on  $\psi^*$  and  $\phi^*$  are independent of  $u_b^*$  and  $\tau_b$ . It follows (since (6.72) and (6.73) are equalities at a solution) that the precise sliding law in this case must be (to first order in  $\sigma$ )

$$u_b^* = C\tau_b^n, \quad (6.74)$$

where  $C = O(1)$  and will depend on the bedrock topography. Note that, in estimating bounds in  $C$ , we may replace  $\int_V dV = \int_0^M dx \int_{\nu h}^{y^*} dy$  by  $\int_0^M dx \int_{\nu h}^\infty dy$ , since the integrands in both (6.72) and (6.73) vanish at large  $y^*$ .

To include the effect of a tangential traction  $\tau_s$ , we make use of the formal approximation  $\nu \ll 1$  and (6.67), which implies that

$$v_t^s = u_b [1 + O(\nu)], \quad (6.75)$$

on  $y = \nu h$ . With this approximation, and defining the average traction by

$$\bar{\tau}_s = \frac{1}{\nu} \frac{1}{M} \int_{S_b} \tau_s dS, \quad (6.76)$$

we find from (6.47), using transformations (6.53), (6.58) and (6.59) and the boundary condition (6.69), that

$$\begin{aligned} u_b^* \tau_b - \frac{1}{n+1} \tau_b^{n+1} \frac{1}{M} \int_V [\tau^{*n+1} - \nu^{n+1}] dV &\leq \frac{n}{n+1} \tau_b u_b^* + \frac{1}{n+1} \bar{\tau}_s u_b^* \\ &\leq \frac{n}{n+1} u_b^{*(n+1)/n} \frac{1}{M} \int_V e^{*(n+1)/n} dV + \bar{\tau}_s u_b^*, \end{aligned} \quad (6.77)$$

whence

$$u_b^* (\tau_b - \bar{\tau}_s) \leq \tau_b^{n+1} \frac{1}{M} \int_V [\tau^{*n+1} - \nu^{n+1}] dV, \quad (6.78)$$

where  $\phi^*$  must now satisfy (6.69), and

$$(1 - \nu^2 h'^2) \phi_{xy}^* + \nu h' (\phi_{yy}^* - \phi_{xx}^*) = \tau_s / \tau_b \quad \text{on } y = \nu h. \quad (6.79)$$

To make (6.79) independent of  $\tau_b$ , we define

$$\tau_s = \tau_b \tau_s^*, \quad (6.80)$$

and therefore

$$\bar{\tau}_s = \tau_b \bar{\tau}_s^* = \tau_b \frac{1}{\nu} \frac{1}{M} \int_{S_b} \tau_s^* dS, \quad (6.81)$$

$$(1 - \nu^2 h'^2) \phi_{xy}^* + \nu h' (\phi_{yy}^* - \phi_{xx}^*) = \tau_s^* \quad \text{on } y = \nu h, \quad (6.82)$$

and

$$u_b^* (1 - \bar{\tau}_s^*) \leq \tau_b^n \frac{1}{M} \int_V (\tau^{*n+1} - \nu^{n+1}) dV. \quad (6.83)$$

The second inequality is seen to be, from (6.77),

$$\tau_b (1 - \bar{\tau}_s^*) \leq u_b^{*1/n} \frac{1}{M} \int_V e^{*(n+1)/n} dV, \quad (6.84)$$

where we require  $\psi^*$  to satisfy (6.63) and (6.64). We may suppose  $\bar{\tau}_s^*$  to be given in (6.83) and (6.84). The inequalities are formally valid as  $\sigma \rightarrow 0$ , but the term  $\bar{\tau}_s^*$  can only be accurate to  $O(\nu)$ . Note that, for a given basal stress, inclusion of a bedrock traction has the effect of reducing the basal velocity, as one would expect. Furthermore, we know that (6.84) is an equality at a solution

of the problem, and also, since  $\psi^*$  satisfies the same constraints as when  $\tau_s^* = 0$ ,  $\int_V e^{*(n+1)/n} dV$  must have the same minimum when the solution is obtained as when  $\tau_s^* = 0$ . From (6.73) and (6.74), this implies that

$$\int_V e^{*(n+1)/n} dV = C^{-1/n}, \quad (6.85)$$

and so the sliding law is of the form

$$u_b^* = C\tau_b^n(1 - \bar{\tau}_s^*)^n,$$

where  $C$  is a function of the bedrock topography. Equation (6.85) can be written in the form

$$\tau_b = Ru_b^{*1/n} + \bar{\tau}_s, \quad (6.86)$$

where  $R = C^{-1/n}$  is a measure of the bedrock roughness; this implies that the basal stress is additively dependent on the ice flow and the bedrock traction, as  $\nu \rightarrow 0$ . Equation (6.85) implies that we must have

$$\tau_s^* < 1, \quad \text{i.e.} \quad \bar{\tau}_s < \tau_b, \quad (6.87)$$

for realistic results. This simply reflects the obvious physical fact that the friction on the bedrock cannot be greater than the applied basal stress. If we are able to impose a basal traction (for example, of frictional type) that is greater than  $\tau_b$ , then (6.86) is not valid, and we must simply have  $u_b^* = 0$  as the proper boundary condition. Of course the above theory is strictly invalid when  $\tau_b - \bar{\tau}_s = O(\sigma)$ , or, more accurately,  $O(\nu^{(n+1)/n})$ , since then (6.86) predicts that  $u_b$ , given by (6.53), is  $O(\sigma)$ .

If  $\sigma/\nu^{n+1} \ll 1$ , (6.50) seems to imply a large (much greater than  $O(1)$ ) basal velocity. Since the velocity of the outer flow was specifically scaled, in (5.20), with  $U_o$  so that it should be  $O(1)$ , this appears to be contradictory. What in fact (6.50) implies is that  $H$  is small, or alternatively that we should not scale the *outer* flow with  $U_o$ , but rather the *basal* flow. If we then denote by  $U_{\text{shear}}$  the scale of the velocity change in the outer flow due to shearing, it is shown by Fowler (1977) that

$$\frac{U_{\text{shear}}}{U_o} \sim \left( \frac{\nu^{n+1}}{\sigma} \right)^{(n+1)/n}, \quad (6.88)$$

and hence the velocity can be written as

$$u = u_b(x) + O[(\nu^{n+1}/\sigma)^{(n+1)/n}]. \quad (6.89)$$

The flow is then effectively one-dimensional and in dynamical studies, the correction term to  $u_b(x)$  would be of less interest. The depth also must be rescaled according to

$$H \sim (\nu^{n+1}/\sigma)^{1/n}. \quad (6.90)$$

For example, if  $\sigma \sim 10^{-2}$  and  $\nu \sim \frac{1}{5}$ , then  $U_{\text{shear}}/U_o \sim 10^{-1}$ . Certain glaciers do indeed appear to have a dominant basal velocity component; for example, eighty per cent of the motion of the Nisqually glacier appears to be due to sliding (Hodge 1974).

## 7. TRIAL ESTIMATES FOR THE ROUGHNESS PARAMETER $R$

In this section, we obtain estimates for the roughness parameter  $R$  by considering particular trial stream and stress functions,  $\psi^*$  and  $\phi^*$ . Even the simplest of such functions involve much computation, and so we restrict ourselves here to the attainment of very crude bounds to illustrate the sort of result that may be obtained. More accurate results would require a specific

description of the bedrock topography, for example. In view of the inadequacy of the model (discussed below in §9), it would be in any case premature to claim great accuracy for such results.

A trial stream function  $\psi^*$  that satisfies (6.63) and (6.64) is

$$\psi^* = y/\nu - h[1 + k_1(y - \nu h)] e^{-k_2(y - \nu h)}, \quad (7.1)$$

and others may be simply written down. The form of  $\psi^*$  in (7.1) is motivated by the flow solution when  $n = 1$  (Newtonian flow), and the form of  $\phi^*$  in (7.2) and (7.22) below is similarly motivated (Fowler 1977). An approximate estimate for  $R$ , by using (7.1) with  $k_1 = 0$ , is established in Appendix A.

Unlike the trial stream function, it is a non-trivial matter even to define a function  $\phi^*$  satisfying the constraints (6.65), (6.66) and (6.69), as well as the periodicity of  $\phi_{xy}^*$ ,  $\phi_{xx}^*$  and  $\phi_{yy}^*$ . The last constraint, in particular, motivates the choice of non-constant  $\alpha$  satisfying (6.70), and not just  $\alpha = 1$ . Although application of the stress variational principle does not require periodicity of the second derivatives of  $\phi^*$ , we expect that satisfaction of this condition will give more accurate estimates, since the actual solution is periodic. We proceed as follows. Let

$$Y = y - \nu h, \quad \theta(x, Y) = \phi^*(x, y), \quad (7.2)$$

so that

$$\left. \begin{aligned} \phi_x^* &= \theta_x - \nu h' \theta_Y, \\ \phi_{xx}^* &= \theta_{xx} - 2\nu h' \theta_{xY} - \nu h'' \theta_Y + \nu^2 h'^2 \theta_{YY}, \\ \phi_{xy}^* &= \theta_{xY} - \nu h' \theta_{YY}, \\ \phi_{yy}^* &= \theta_{YY}; \end{aligned} \right\} \quad (7.3)$$

the transformation (7.2) shifts the boundary to  $Y = 0$ . The zero shear stress constraint there, (6.66), becomes

$$\nu h' (\theta_{YY} - \theta_{xx} + 2\nu h' \theta_{xY} + \nu h'' \theta_Y - \nu^2 h'^2 \theta_{YY}) + (1 - \nu^2 h'^2) (\theta_{xY} - \nu h' \theta_{YY}) = 0,$$

$$\text{i.e.} \quad \theta_{xY} = (\nu h' / [1 + \nu^2 h'^2]) [\theta_x - \nu h' \theta_Y]_x \quad \text{on} \quad Y = 0. \quad (7.4)$$

The conditions (6.65) and (6.69) are satisfied by  $\phi^*$ , provided

$$\theta \sim -\nu y \int^x \alpha(x) dx = -\nu Y \int^x \alpha(x) dx - \nu^2 h \int^x \alpha(x) dx, \quad Y \rightarrow \infty. \quad (7.5)$$

$$\text{Let us define} \quad \theta_{xY} = \nu f'(x) \quad \text{on} \quad Y = 0, \quad (7.6)$$

$$\text{so that} \quad \theta_Y = \nu f(x) \quad \text{on} \quad Y = 0; \quad (7.7)$$

substituting (7.7) into (7.4), we have

$$f'(x) = [h' / (1 + \nu^2 h'^2)] [\theta_x - \nu^2 h' f(x)]_x \quad \text{on} \quad Y = 0. \quad (7.8)$$

$$\text{Let } g(x) \text{ be defined by} \quad f'(x) = h' g'(x) / (1 + \nu^2 h'^2); \quad (7.9)$$

$$\text{then (7.8) implies} \quad \theta_x = \nu^2 h' f(x) + g(x) \quad \text{on} \quad Y = 0, \quad (7.10)$$

and so (6.66) will be satisfied by  $\theta$  provided we choose

$$\theta = k(x), \quad \theta_Y = \nu f(x) \quad \text{on} \quad Y = 0, \quad (7.11)$$

where  $k$  and  $f$  satisfy

$$\left. \begin{aligned} k'(x) &= \nu^2 h'(x) f(x) + g(x), \\ f'(x) &= h'(x) g'(x) / (1 + \nu^2 h'^2). \end{aligned} \right\} \quad (7.12)$$

A particular (and obvious) choice is to define

$$\theta = \left[ \begin{aligned} k(x) + \nu^2 h \int_0^x \alpha(x) dx & F(Y) - \nu Y \int_0^x \alpha(x) dx - \nu^2 h \int_0^x \alpha(x) dx, \\ f(x) &= - \int_0^x \alpha(x) dx, \end{aligned} \right] \quad (7.13)$$

where  $\alpha$  is to satisfy (6.70), and we require  $F(Y)$  to satisfy

$$F(0) = 1, \quad F'(0) = 0, \quad F(\infty) = 0. \quad (7.14)$$

Once again, motivated by the Newtonian solution (Fowler 1977), we put

$$F(Y) = (1 + kY) e^{-kY}, \quad (7.15)$$

which satisfies (7.14). In addition,  $\phi_{xx}^*$ ,  $\phi_{xy}^*$  and  $\phi_{yy}^*$  are periodic if  $\left[ k(x) + \nu^2 h \int_0^x \alpha(x) dx \right]$  is periodic.

Now  $f' = -\alpha$  is periodic; therefore  $k' - \nu^2(hf)'$ , and hence  $k' - \nu^2 h'f$  (since  $hf'$  is periodic) must be periodic. From (7.12), we therefore require  $g$  to be periodic. A simple but useful choice of  $g$  is

$$g(x) = ah(x) + c, \quad (7.16)$$

where  $a$  and  $c$  are constants to be chosen. From (7.12),

$$f(x) = - \int_0^x \alpha(x) dx = a \int_0^x \frac{h'^2}{1 + \nu^2 h'^2} dx, \quad (7.17)$$

whence we choose, from (6.70),

$$a = -M \left( \int_0^M \frac{h'^2 dx}{1 + \nu^2 h'^2} \right)^{-1}. \quad (7.18)$$

Now we can write the first equation in (7.12) as

$$k' = \nu^2(hf)' - \nu^2 h'f' + ah + c,$$

whence

$$k(x) = \nu^2 hf - \int_0^x \frac{a\nu^2 h h'^2}{1 + \nu^2 h'^2} dx + a \int_0^x h dx + cx \quad (7.19)$$

(choosing the constant of integration as zero). We now ensure that  $k - \nu^2 h'f$  is periodic by choosing (since  $h$  has zero mean)

$$c = \frac{\nu^2 a}{M} \int_0^M \frac{h h'^2}{1 + \nu^2 h'^2} dx, \quad (7.20)$$

so that  $k(x)$  is given by

$$k(x) = -M \left( \int_0^M \frac{h'^2 dx}{1 + \nu^2 h'^2} \right)^{-1} \left[ \int_0^x h dx + \nu^2 \left( \frac{x}{M} \int_0^M \frac{h h'^2 dx}{1 + \nu^2 h'^2} - \int_0^x \frac{h h'^2 dx}{1 + \nu^2 h'^2} + h \int_0^x \frac{h'^2 dx}{1 + \nu^2 h'^2} \right) \right]. \quad (7.21)$$

With this definition, and that of  $F(Y)$  in (7.15), the trial function  $\theta$  in (7.13) is given by

$$\theta = -\frac{1}{h_*^2} \left[ \int_0^x h dx (1 + kY) e^{-kY} + \nu Y \int_0^x h'^2 dx + O(\nu^2) \right], \quad (7.22)$$

where  $h_*$  is the scaled mean quadratic bedrock slope defined by

$$h_* = \left( \frac{1}{M} \int_0^M h'^2 dx \right)^{\frac{1}{2}}. \quad (7.23)$$

The function  $\theta$  defined by (7.22) is used in Appendix A to determine a lower bound for the roughness parameter  $R$  (to  $O(1)$ ).

The results are naturally critically dependent on the assumed form of the bedrock roughness profile. If we suppose only that  $|h''| \leq 1$  (as well as  $|h|, |h'| \leq 1$ ), then we find that, to leading order in  $\nu$ ,

$$0.55(2h_*^2)^{\frac{1}{3}} \leq R \leq 2.76, \quad (7.24)$$

where we have taken  $n = 3$  in Glen's flow law. This assumption on  $h$  is really one of physical *smoothness*, and implies essentially that (as already mentioned) bedrock roughness is effectively absent at short wavelengths; this condition is *not* met by the idealized 'white bedrock' for example (Kamb 1970).

As a specific example, we consider a sinusoidal bedrock  $h = \cos x$ ; we then obtain (for  $n = 3$ ) the leading-order bounds

$$1.39 \leq R \leq 1.53, \quad (7.25)$$

which are as accurate as could be wished for in view of the defects of the present model.

## 8. EFFECTS OF REGELATION

As previously mentioned, putting  $\alpha = 0$  in the ice flow model has the physical interpretation that regelation is negligible, except over the smallest obstacles; thus we expect  $\alpha = 0$  to be a valid approximation, provided roughness is typically absent at wavelengths of the order of 1 mm. We can see this more clearly as follows. If we consider the functional  $J$  defined by (6.12) with  $\tau_s = 0$ , but with the integral on  $S_b$  replaced by

$$-\int_{S_b} \left[ \sigma_{nn} \left( v_n - \alpha \frac{\partial T}{\partial n} \right) + \alpha T \frac{\partial T}{\partial n} \right] dS - \frac{1}{2} \alpha \int_{V_b} |\nabla T|^2 dV, \quad (8.1)$$

where  $V_b$  is the bedrock volume below  $S_b$ , then it is easy to see from (5.96) (with  $r = \infty$ ) that the first variation  $\delta J = 0$  at a solution. Furthermore, the equivalent of (6.39) is (to leading order)

$$J_0 = \frac{n}{n+1} M \nu \tau_b u_b - \frac{1}{2} \alpha \int_{S_b} T \frac{\partial T}{\partial n} dS, \quad (8.2)$$

and the regelation adds a (positive) component  $-\frac{1}{2} \alpha \int_{S_b} T (\partial T / \partial n) dS$  to the functional. It is clear that regelation is only of importance if

$$\left| \alpha \int_{S_b} T \frac{\partial T}{\partial n} dS \right| \sim \nu, \quad (8.3)$$

and the only way this can be achieved is if  $\partial T / \partial n$  becomes of order  $\nu / \alpha$ , which is essentially if

$$|h''| \sim \nu / \alpha, \quad (8.4)$$

since if roughness is present on this scale then  $p$  (and hence  $T$ ) may change by  $O(1)$  from the upstream to the downstream side of very small-scale obstacles.

If such roughness is present (as seems unlikely), there is no obvious way in which to measure the regelative component of the drag (which would require a detailed knowledge of the bedrock over very small scales), although an estimated drag  $\tau_s$  could be incorporated, as in §6. However, until the necessity for considering such a complication has been shown, it seems more sensible to proceed on the rational assumption that  $\alpha = 0$ , and thus neglect regelation altogether.

## 9. PHYSICAL EFFECTS

In this section we consider some of the physical effects likely to hinder any sensible comparison between measured basal velocities and predicted ones. The intention is to fulfil Robin's (1976) hope that theoreticians will 'pause in developing more complex mathematical models of sliding in order to study whether or not the assumptions on which their theories are based involve an adequate description of conditions and processes in the basal layers of glaciers'. Robin's comment was concerned particularly with the possibility that cold patches could occur at the ice-rock interface. In the model presented here, such cold patches can arise naturally if the film thickness  $\Sigma \rightarrow 0$ , or (necessarily) if the sliding is of subtemperate type. Similarly, although cavitation is not considered in this paper, it can easily be incorporated into the formulation of the problem. What is felt to be the most crucial quantitative assumption is that the flow law for temperate ice is independent of the moisture content, and that no consideration is made of the hydrology of the basal flow layer. As observed by Carol (1947), basal ice can have a vastly different structure upstream and downstream of obstacles, and so it is likely that neglect of moisture effects is at the least a major source of inaccuracy in determining the sliding law.

It is further not clear what the effect of the presence of moisture will be on the amount of regelation taking place. In one instance, Kamb (1970) observed 'massive amounts of regelation on a scale of about 35 cm'; there seems to be no obvious explanation of this. Robin ascribed it to the importance of convective heat transport at larger velocities; this corresponds to taking a finite value of  $r$  in (5.96), but there does not seem to be any alteration in the magnitude of  $\alpha$ . It is not correct to say that neglect of temperature convection in the energy equation has an effect either: for temperate ice, the temperature is *defined* by the Clausius-Clapeyron equation, and the energy equation determines the moisture content, *not* the temperature.

Thus, to accommodate these observations, it is necessary to account for moisture transport through the ice, and also the effect of this on the flow law. Some discussion of this was given in I, but it would seem that incorporation of these features into a detailed mathematical analysis of sliding must await a satisfactory model of the processes involved. The provision of a suitable transport equation for moisture is precisely the quantitative description of the 'heat pump' effect called for by Robin (1976).

Let us now turn to some practical considerations that should be of relevance to field studies. As shown by Nye (1952), the dimensional basal stress is given by

$$\tau_b = \rho g h_1 \sin \chi, \quad (9.1)$$

where  $h_1$  is the depth measured perpendicularly to the line of mean slope, and  $\chi$  is the inclination of the surface to the horizontal. Since, in a dimensionless formulation (for example, I), the formula (9.1) implicitly assumes that  $\chi$  varies over length scales of the order of the glacier length, it follows that, as suggested by Hodge (1974),  $\chi$  should be measured by taking the slope between two points on the surface reasonably far apart (i.e. hundreds of metres) since, if estimated over smaller distances, errors may be introduced owing to relatively small-scale variations in the bedrock profile (in much the same way as Robin (1967) obtained *corrections* to the surface slope of the Antarctic ice sheet). In then attempting to correlate measured values of  $\tau_b$  and  $u_b$ , it is most important to realize that in regions where subtemperate sliding is occurring, that is, where

$$u_b = F[\tau_b, T] \quad (9.2)$$



(Fowler 1979), and  $T$  is close to, but not quite at, the pressure melting temperature, there will be no apparent correlation between  $\tau_b$  and  $u_b$ . The difficulties of determining from temperature measurements alone the location of such subtemperate basal zones would seem at this time to be insuperable, and indirect observation may have to be sufficient.

## 10. CONCLUSIONS

In this paper we have presented an analysis of glacier sliding that takes account of the non-linear dependence on stress of Glen's flow law. The analysis is made possible by a scaling of the problem that shows, for roughness scales of about 1 m, that regelation can be effectively neglected *from the point of view of the ice flow*. This analysis should thus be seen as an extension to nonlinear flow laws of Nye's earlier work (1969). While the nonlinearity has not been previously taken into account, it is worth noting some of the physical mechanisms that may in future work substantially alter the quantitative conclusions presented here.

We have already discussed the possible effect of moisture production within temperate ice; equally important might be the porosity of the bedrock material, which could substantially affect the degree of sliding by control of the basal melt-water régime (Chadbourne *et al.* 1975). This last article compared experimental measurements of sliding ice with theoretical results from Nye's regelation-based theory, and found large discrepancies. In the related problem of regelating flow past cylindrical wires, Drake & Shreve (1973) found similar discrepancies, and suggested several physical mechanisms that might explain these. Most notably, the formation of a trace at a driving stress of *ca.* 1 bar is consistent with the formation of cavities when the triple-point pressure is reached, and is associated with a definite transition from low to high velocities. Secondly, the presence of solutes in the water film can affect the melting temperature. An analysis of the effect of these on glacial sliding was made by Hallet (1976). In particular, he showed that the transition wavelength  $\lambda_*$  given by (5.59) is decreased by the presence of solutes; on the other hand, solutes may contribute to short-wavelength roughness components, for example so-called solutional furrows.

Other phenomena discussed by Drake & Shreve (1973), as well as by Nye (1973), are the Frank (1967) instability of the ice-water interface, and the supercooling of ice that is necessary to freeze water at a finite rate. Additionally, there is the possibility of cavities and cold patches occurring when the water film thickness described by (5.69) becomes infinite or zero.

None of these topics is included in the model described here. It will be seen that all of them are closely associated with the process of regelation which, as we have said, is peripheral to the aim of this work. It is difficult to see how regelation could be combined with the nonlinear flow law in obtaining solutions. One possible way is to use the fact that  $\alpha \ll 1$  to consider the 'microscopic' regelative flow as exerting a boundary-layer type effect on the 'macroscopic' nonlinear flow round the larger obstacles. It might then be possible to treat the flow as locally Newtonian if the longitudinal stress were small enough (and ice were Newtonian at small stresses). There are of course grounds for arguing that the complexity of such a theoretical process is not justified in practice, but that is a different question.

Having made a specific set of physical assumptions on the nature of the ice flow, our subsequent analysis is rational in the sense of I, i.e. no further arbitrary assumptions on the nature of the solution are made, and all approximations are carried out in a consistent, asymptotic fashion.

In this case, there exist complementary variational principles for the ice flow, which determine

bounds on the drag induced by a non-zero basal velocity. It should be emphasized that these principles apply when  $n \neq 1$  in Glen's flow law, and in addition the assumption is *not* made that the bedrock 'corrugation'  $\nu$  is vanishingly small. The nature of the obtained bounds indicates that the flow law is of the dimensionless form

$$u_b = \frac{\sigma}{\nu^{n+1}} \left( \frac{\tau_b}{R} \right)^n, \quad (10.1)$$

and we obtain bounds on  $R$ . The expression (10.1) is valid provided  $\nu^{n+1} \ll 1$ , which *formally* requires  $\nu \ll 1$ , although  $\nu \lesssim \frac{1}{3}$  is quite sufficient to obtain a non-trivial sliding velocity. The constant  $R$  depends on the (scaled) bedrock topography  $h(x)$ , and also on the corrugation  $\nu$ , but (by construction) has a finite limit of order one as  $\nu \rightarrow 0$ , and so we *now* seek bounds on  $R$  as  $\nu \rightarrow 0$ , as this will give us a reasonable estimate of its actual value. For the specific case of a sinusoidal bedrock  $h = \cos x$  (for which  $R = 1$  when  $n = 1$  as  $\nu \rightarrow 0$ ), we obtain

$$1.39 \leq R \leq 1.53 \quad (10.2)$$

when  $n = 3$ ,  $\nu \rightarrow 0$ .

In previous work, it has been questioned whether sliding without cavitation is capable of predicting basal velocities of the observed order of magnitude. We can state first that small sliding velocities may always be explained by supposing that the sliding is subtemperate. Larger (steady-state) velocities can be explained on the basis that  $\nu$  is sufficiently small (so that  $\sigma/\nu^{n+1}$  is large), since  $\sigma/\nu^{n+1}$  really represents the proportion of the total motion due to sliding; in the steady state the total motion is governed by the surface accumulation rate, and is nothing to do with the sliding law.

Much larger (non-steady-state) sliding velocities of the order of kilometres per year, associated with surging glaciers, can only be explained by a sliding law that changes rapidly with basal stress at some critical point, or which is even multivalued. The obvious (and apparently only) phenomenon capable of producing such a violent change of behaviour is sub-glacial cavitation (Lliboutry 1968), and preliminary work indicates that it may indeed have such a crucial effect (Fowler 1977). (This is not to say that cavitation is the only mechanism for generating surges. There is also the possibility that these are due to enhanced shearing in temperate ice zones due to 'runaway' of the moisture content associated with a thermal instability of the glacier (Robin 1955; Clarke *et al.* 1977).) With regard to large velocities, notice that if  $\sigma/\nu^{n+1} \gg 1$ , it is quite possible that the full temperate sliding velocity may never be reached, and all sliding is then of subtemperate type; this may be true even under an otherwise fully temperate glacier, and in this case the title question of Robin's (1976) paper may be definitely answered: no.

The model presented here is deficient in one major respect, and that is that no theoretical treatment of the moisture transport through the ice is offered. This is beyond the scope of the paper, but it is hoped that a satisfactory treatment of this will soon be forthcoming; until this is done, no real confidence should be placed in the precise numerical and analytical results given here.

I should like to thank both referees for their comments and suggestions concerning this paper.

## APPENDIX A

In this appendix we obtain explicit leading-order bounds for the roughness parameter  $R$ . In view of the complexity of the necessary calculations, we restrict ourselves to consideration of the particular limit  $\nu \rightarrow 0$ . Since a precise estimation of  $R$  could only be made if the bedrock profile  $h$  were accurately known, we here consider two types of bedrock. (i) We make *no* assumptions on the profile  $h$ , save that

$$|h''| \leq 1 \quad (\text{A } 1)$$

(we can assume  $|h|, |h'| \leq 1$  by choice of the scales  $[x]$  and  $[y]$ ). (A 1) seems a fairly realistic assumption to make; it essentially means that roughness is absent at short wavelengths (as we implicitly assume) and is *not* compatible with the notion of a *white bedrock*, as defined for example by Lliboutry (1976), which would have unbounded values of  $|h''|$ .

(ii) We consider a sinusoidal bedrock  $y = \nu \cos x$ , which shows that useful bounds can indeed be obtained. It should be emphasized that in each example the bounds obtained are by no means the best available.

$$\text{Type (i): } |h''| \leq 1.$$

We define

$$y - \nu h = Y, \quad (\text{A } 2)$$

and consider the trial function

$$\psi^* = y/\nu - h e^{-kY}, \quad (\text{A } 3)$$

which satisfies the constraints of §6. We then find from (6.72) that, to leading order in  $\nu$ ,

$$\begin{aligned} R &\leq \frac{1}{M} \int_0^M \int_0^\infty [(k^2 h - h'')^2 + 4k^2 h'^2]^{\frac{1}{2}(n+1)/n} e^{-[(n+1)/n] kY} dY dx \\ &\leq \int_0^\infty [(1+k^2)^2 + 4k^2]^{\frac{1}{2}(n+1)/n} e^{-[(n+1)/n] kY} dY \\ &= \frac{n}{(n+1)k} [(1+k^2)^2 + 4k^2]^{\frac{1}{2}(n+1)/n}. \end{aligned} \quad (\text{A } 4)$$

One easily finds that (A 4) is a minimum when

$$k = \{([n(n+2) + 9]^{\frac{1}{2}} - 3)/(n+2)\}^{\frac{1}{2}}, \quad (\text{A } 5)$$

and therefore

$$R \leq \frac{n}{n+1} \left\{ \frac{[9 + n(n+2)]^{\frac{1}{2}} - 3}{n+2} \right\}^{-\frac{1}{2}} \left\{ 2 \frac{n+1}{(n+2)^2} \{3[9 + n(n+2)]^{\frac{1}{2}} + n - 7\} \right\}^{\frac{1}{2}(n+1)/n}. \quad (\text{A } 6)$$

Putting  $n = 3$ , we find

$$R \leq 2.76; \quad (\text{A } 7)$$

for a Newtonian fluid,  $n = 1$ , and the result is

$$R \leq 2.48. \quad (\text{A } 8)$$

We now use the trial function for the stress principle given by (7.22). To leading order in  $\nu$ , we then find that

$$R^{-n} \leq \frac{1}{M} \int_0^M \int_0^\infty [K'^2 F'^2 + \frac{1}{4}(KF'' - K''F)^2]^{\frac{1}{2}(n+1)} dx dY, \quad (\text{A } 9)$$

where

$$K(x) = -\frac{1}{h_*^2} \int_0^x h dx, \quad F(Y) = (1+kY) e^{-kY}. \quad (\text{A } 10)$$

Substituting (A 10) into (A 9), and using the assumptions on  $h$ , we find that

$$R^{-n} \leq \frac{1}{(2h_*^2)^{n+1}} \int_0^\infty [4k^4 Y^2 + (1+kY+k^2|1-kY|)^2]^{\frac{1}{2}(n+1)} e^{-(n+1)kY} dY. \quad (\text{A } 11)$$

Unfortunately, the integral in (A 11) is non-trivial to evaluate: we obtain a crude estimate as follows. From (A 11) we have

$$\begin{aligned} R^{-n} &< \frac{1}{(2h_*^2)^{n+1}} \int_0^\infty [4k^2(1+kY)^2 + (1+k^2)^2(1+kY)^2]^{\frac{1}{2}(n+1)} e^{-(n+1)kY} dY \\ &= \frac{1}{(2h_*^2)^{n+1}} \frac{e^{n+1} (1+6k^2+k^4)^{\frac{1}{2}(n+1)}}{k (n+1)^{n+2}} \Gamma(n+2, n+1), \end{aligned} \quad (\text{A } 12)$$

where  $\Gamma(\alpha, x)$  is the incomplete  $\Gamma$ -function defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt \quad (\text{A } 13)$$

(Abramowitz & Stegun 1968). The minimum value of (A 12) is when

$$k^2 = [(9n^2 + 2n + 1)^{\frac{1}{2}} - 3n]/(2n + 1), \quad (\text{A } 14)$$

and after some algebra we find that (A 12) may be written as

$$R > 0.55[2h_*^2]^{\frac{4}{3}}, \quad n = 3, \quad (\text{A } 15)$$

the equivalent result for a Newtonian fluid being

$$R > 0.16[2h_*^2]^2, \quad n = 1. \quad (\text{A } 16)$$

Notice that the mean quadratic slope  $h_*$  appears explicitly in this bound. Summing up, we have, for  $|h''| \leq 1$ ,

$$\left. \begin{aligned} 0.55[2h_*^2]^{\frac{4}{3}} < R < 2.76, \quad n = 3, \\ 0.16[2h_*^2]^2 < R < 2.48, \quad n = 1. \end{aligned} \right\} \quad (\text{A } 17)$$

*Type (ii):*  $h = \cos x$

We choose

$$\psi = y/\nu - h(1+Y)e^{-Y}, \quad (\text{A } 18)$$

so that to leading order

$$\begin{aligned} R &\leq \frac{1}{M} \int_0^M \int_0^\infty (4h^2 Y^2 e^{-2Y} + 4h'^2 Y^2 e^{-2Y})^{\frac{1}{2}(n+1)/n} dx dY \\ &= 2^{(n+1)/n} \int_0^\infty Y^{(n+1)/n} e^{-[(n+1)/n]Y} dY \\ &= 2^{(n+1)/n} \left(\frac{n}{n+1}\right)^{(2n+1)/n} \Gamma\left(\frac{2n+1}{n}\right) \\ &= \left(\frac{2n}{n+1}\right)^{(n+1)/n} \Gamma\left(\frac{n+1}{n}\right), \end{aligned} \quad (\text{A } 19)$$

where  $\Gamma(z)$  is the gamma function. For  $n = 3$ , this gives

$$R \leq 1.53, \quad (\text{A } 20)$$

whereas, for  $n = 1$ , we find

$$R \leq 1. \quad (\text{A } 21)$$

(In fact it is known that  $R = 1$  for  $\nu \rightarrow 0$  and  $n = 1$  (Fowler 1977), and so the choice (A 18) is optimal in this sense. This is, in fact, why the trial function was chosen in the form (A 18).)

For the trial stress function, we similarly choose that given by (7.22), with  $k = 1$ ; then we obtain

$$\begin{aligned} R^{-n} &\leq \frac{1}{h_*^{2(n+1)}} \int_0^\infty Y^{n+1} e^{-(n+1)Y} dY \\ &= \frac{1}{h_*^{2(n+1)}} \frac{\Gamma(n+2)}{(n+1)^{n+2}}, \end{aligned} \quad (\text{A } 22)$$

$$\text{and thus (for integral } n) \quad R \geq [(n+1) h_*^2]^{(n+1)/n} / (n!)^{1/n}; \quad (\text{A } 23)$$

$$\text{putting } n = 3 \text{ and } h_* = \frac{1}{\sqrt{2}}, \text{ we find} \quad R \geq 1.39. \quad (\text{A } 24)$$

For  $n = 1$ , we find  $R \geq 1$  so that (as expected)  $R = 1$  when  $n = 1$ . For  $n = 3$ , the two inequalities combine to give

$$1.39 \leq R \leq 1.53. \quad (\text{A } 25)$$

#### APPENDIX B

Let us suppose that the flow law takes the general form

$$\left. \begin{aligned} e_{ij} &= (f(\tau)/\tau) \tau_{ij}, \\ e &= f(\tau), \end{aligned} \right\} \quad (\text{B } 1)$$

rather than the Glen's law form (5.93). For Glen's law we would choose  $f(\tau)$  proportional to  $\tau^n$ .

We also define  $g$ , the inverse of  $f$ , as

$$\left. \begin{aligned} \tau &= g(e), \\ \tau_{ij} &= (g(e)/e) e_{ij}. \end{aligned} \right\} \quad (\text{B } 2)$$

$$\text{Consider the functions} \quad \Gamma(e) = \int_0^e g(e) de, \quad \hat{\Gamma}(\tau) = \int_0^\tau f(\tau) d\tau. \quad (\text{B } 3)$$

Using (6.29), we find

$$\frac{\partial \Gamma}{\partial e_{ij}} = \frac{1}{2} \frac{g(e)}{e} e_{ij} = \frac{1}{2} \tau_{ij}, \quad \frac{\partial \hat{\Gamma}}{\partial \tau_{ij}} = \frac{1}{2} \frac{f(\tau)}{\tau} \tau_{ij} = \frac{1}{2} e_{ij}. \quad (\text{B } 4)$$

Thus these choices of  $\Gamma$  and  $\hat{\Gamma}$  satisfy the potential requirements (6.9) and (6.21). Furthermore, at a solution where (B 1) and (B 2) are valid,

$$\begin{aligned} \Gamma &= \int_0^e g(e) de = \int_0^\tau \tau f' d\tau \\ &= [\tau f]_0^\tau - \int_0^\tau f(\tau) d\tau \\ &= \tau e - \hat{\Gamma} \end{aligned}$$

(we require that  $f(0) = g(0) = 0$ ); also from (B 2)

$$\frac{1}{2} \tau_{ij} e_{ij} = g(e) e = \tau e.$$

Hence,

$$\Gamma + \hat{\Gamma} = \frac{1}{2} \tau_{ij} e_{ij} \quad (\text{B } 5)$$

at a solution, thus satisfying (6.24).

It follows that the choices of  $\Gamma$  and  $\hat{\Gamma}$  in (B 3) provide variational principles for the functionals  $J$  and  $\mathcal{H}$  given by (6.12) and (6.22), and that at a solution these functionals have the same value. If we further have  $\Gamma$  and  $\hat{\Gamma}$  convex, in the sense that  $J_2$  defined by (6.41) is positive and  $\mathcal{H}_2$  defined by (6.44) is negative, then the functionals  $\mathcal{H}_\tau$  and  $J_\vee$  provide upper and lower bounds for  $J_0$ , (6.46).

To obtain the equivalent of (6.47), we must try to find  $\int \Gamma dV$  in terms of the drag. Specifically, if we ignore inertia terms  $f_i$  and set  $\tau_s = 0$ , then (6.40) implies

$$J_V = \int_V \Gamma dV, \quad (\text{B } 6)$$

whereas (6.43) implies 
$$\mathcal{H}_\tau = - \int_V \hat{\Gamma} dV + \nu \tau_b u_b M. \quad (\text{B } 7)$$

The bounds on  $J_0$  can thus be written

$$- \int_V \Gamma dV + \nu \tau_b u_b M \leq \left( \int_V \Gamma dV \right)_{\text{soln}} \leq \int_V \Gamma dV, \quad (\text{B } 8)$$

where the suffix soln indicates the value taken at the solution of the problem.

Since for the power law (and also polynomial laws, see below),  $\left( \int_V \Gamma dV \right)_{\text{soln}} < \nu \tau_b u_b M$ , these bounds give two bounds on  $u_b$ , in the form

$$\left. \begin{aligned} \nu M \tau_b u_b - \left( \int_V \Gamma dV \right)_{\text{soln}} &\leq \int_V \hat{\Gamma} dV, \\ \left( \int_V \Gamma dV \right)_{\text{soln}} &\leq \int_V \Gamma dV. \end{aligned} \right\} \quad (\text{B } 9)$$

Here, both sides are positive; the right sides of (B 9) are trial estimates, and hence we wish to evaluate  $\left( \int_V \Gamma dV \right)_{\text{soln}}$  in terms of the drag  $\tau_b$ . Observe that, at a solution,

$$\int_V (\Gamma + \hat{\Gamma}) dV = \int_V \frac{1}{2} \ell_{ij} \tau_{ij} dV = \nu M \tau_b u_b, \quad (\text{B } 10)$$

by using the same argument as in (6.37). Thus, it is equally useful to evaluate  $\left( \int_V \hat{\Gamma} dV \right)_{\text{soln}}$ . Now,

$$\begin{aligned} \nu M \tau_b u_b &= \left( \int_V \frac{1}{2} \ell_{ij} \tau_{ij} dV \right)_{\text{soln}} = \left( \int_V \tau f(\tau) dV \right)_{\text{soln}} \\ &= \left( \int_V \tau \hat{\Gamma}'(\tau) dV \right)_{\text{soln}}. \end{aligned} \quad (\text{B } 11)$$

Thus, we wish to evaluate  $\left( \int_V \hat{\Gamma} dV \right)_{\text{soln}}$  in terms of  $\left( \int_V \tau \hat{\Gamma}' dV \right)_{\text{soln}}$ . For the particular case of a power-law fluid, one is simply a multiple of the other. For more general models, we must seek reasonable bounds. For an Ellis model fluid (Hopke & Slattery 1970; Thompson 1979; Hutter 1980), we write

$$e = K_1 \tau + K_2 \tau^n, \quad (\text{B } 12)$$

where  $K_1$  and  $K_2$  are constants, and  $n$  is the flow-law parameter ( $n \approx 3$ ). Then

$$\hat{\Gamma} = \int_0^\tau e d\tau = \frac{1}{2} K_1 \tau^2 + K_2 \tau^{n+1} / (n+1), \quad (\text{B } 13)$$

whereas 
$$\tau \hat{\Gamma}' = K_1 \tau^2 + K_2 \tau^{n+1}. \quad (\text{B } 14)$$

It is clear that for a polynomial flow law with positive coefficients, bounds can easily be obtained. For example, the form (B 12) implies that

$$2\hat{\Gamma} \leq \tau \hat{\Gamma}' \leq (n+1)\hat{\Gamma}, \quad (\text{B } 15)$$



provided  $n > 1$ . It follows that

$$[1/(n+1)] \tau \hat{I}' \leq \hat{I} \leq \frac{1}{2} \tau \hat{I}'^n, \quad (\text{B } 16)$$

and hence, from (B 11), 
$$\frac{\nu M \tau_b u_b}{n+1} \leq \left( \int_V \hat{I} dV \right)_{\text{soln}} \leq \frac{1}{2} \nu M \tau_b u_b. \quad (\text{B } 17)$$

From (B 10) we have 
$$(n/(n+1)) \nu M \tau_b u_b \geq \left( \int_V \Gamma dV \right)_{\text{soln}} \geq \frac{1}{2} \nu M \tau_b u_b \quad (\text{B } 18)$$

(thus confirming the statement after (B 8)). Hence, (B 9) gives

$$\frac{\nu M \tau_b u_b}{n+1} \leq \int_V \hat{I} dV, \quad (\text{B } 19)$$

$$\frac{1}{2} \nu M \tau_b u_b \leq \int_V \Gamma dV, \quad (\text{B } 20)$$

as the two bounds for the sliding law. The only difference between these bounds and those for Glen's law is the factor  $\frac{1}{2}$  in (B 20), rather than the value  $n/(n+1)$ .

For practical purposes, the correction of Glen's law is only of use if  $\tau \rightarrow 0$  in the flow. While this is so near the surface of an ice flow, in the present case it is not so, since  $\tau_2$  is finite (non-zero) at  $y \rightarrow \infty$ , and  $\tau_1$  is mostly non-zero near the base of the flow (although  $\tau_2 \rightarrow 0$  there). In any case, the change is not a large one. Notice that the bound (B 19) corresponds (see (6.73)) to an upper bound of the flow velocity, and this is unchanged by the Newtonian viscosity at smaller stresses. Also, it is clear that the bounds (B 19) and (B 20) are applicable to any polynomial law  $f(\tau)$  of highest degree  $n$ , provided all powers of  $\tau$  in  $f(\tau)$  are greater than or equal to one; this is a necessary condition that the viscosity is not zero at zero stress. Further, equation (6.3) of Johnson (1961) shows, after a little manipulation, that the convexity of  $\Gamma$  and  $\hat{I}$  (and hence the validity of the maximum and minimum principles) is attained if

$$f' > 0, \quad (\text{B } 21)$$

i.e. if the strain rate increases with stress. This is certainly so for polynomials of degree greater than 1, with positive coefficients, as well as for other types of law, for example,  $f(\tau)$  proportional to  $\sinh a\tau$ . One drawback, however, is that application of the second inequality (B 20) requires an inversion of the function  $f$  to determine  $g$ . This is not generally explicitly possible, even for the simple Ellis model.

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